

# **MECHANICS**

**for B.Sc. Students**

**According to New Syllabus**

***Ross Nazir Ullah***

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## Preface

This textbook is intended for B.Sc. Physics students. The material in this book deals with fundamental topics in Mechanics. In writing 'Mechanics' as a basic textbook, I have striven to make it a self explanatory book. My major objective has been to provide the student with a clear and logical approach to the basic principles of Physics. Emphasis is placed on presenting the concepts and applications of Physics in a precise, but realistic fashion.

It has been my experience that many students struggle through such a course for a variety of reasons such as an inadequate and non-appropriate guidance. I have tried to keep such students in mind by introducing basic concepts with mathematical formulas. I have included some illustrative examples, which should assist the student in understanding the concepts, and could provide some basis for solving unsolved problems.

I have spent a lot of time and effort in writing this textbook. I hope it will help you to teach and learn Mechanics. In turn you can help me by telling me to know what still needs to be improved.

Please give your suggestions and criticisms. They are welcomed and will be taken into consideration for next edition.

October 1, 1999.

Ross Nazir Ullah

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## Introduction

This textbook of 'Mechanics' contains many features, which should benefit to the student. I feel it is appropriate to offer some words of advice to understand the material in the text. These comments are based upon my 22 years of experience in teaching at college level.

You should read the text carefully for obtaining a good understanding of the subject. It is not possible for an ordinary student to absorb the full meaning of scientific writing after one reading. Several readings of the text supplemented with class lectures are necessary. Your level of understanding will increase with your efficient study habits, discussion with your class fellows and asking questions from your teachers.

The lectures will be much more meaningful if you read the corresponding text in advance. You should try to avoid the practice of delaying study until few days left before the exam. For further guide please read, 'How to succeed in Physics' in my supplementary book.

R. P. Feynman, Nobel prize winner in Physics said, 'You do not know anything until you have practiced'. So you must develop the ability to solve the problems.

It is my hope that this text will be helpful and beneficial for you. Start learning Mechanics with a positive attitude.

With best of luck.

  
(Ross Nazir Ullah)

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## 1-1: Force Laws

Galileo found a description of how bodies fall. Within 50 years it was superseded a deeper insight into nature with Newton's universal law of gravity. Newton found the force of gravity as a fundamental force of nature. This force of gravity holds together planets and stars, organizes solar systems and galaxies.

$$F = -G (m_1 m_2 / r^2) r \quad \dots(1)$$

In 18<sup>th</sup> century Charles Augustin Coulomb experimentally found that electric force is similar to gravity. It is known as Coulomb's law:

$$F = K_e (q_1 q_2 / r^2) r \quad \dots(2)$$

Maxwell (1831-1879) succeeded in unifying electric and magnetic forces, into one electromagnetic force.

With the 20<sup>th</sup> century, physicists realized that neither gravity nor electromagnetism held the compact nucleus together, but that a new force – the strong force, it overcomes the electric repulsion between protons and holds the nucleus together. Unlike gravity and electricity, the strong force does not extend to distant corners of the universe; it has limited range – the size of a nucleus,  $10^{-13}$  cm.

Natural radioactivity could not be explained by any of the known force – strong, electromagnetic, or gravitational. Another force was responsible for the decays of nuclei – the weak force. This is also short range force, i.e., up to the size of a nucleus. Yet the weak force causes some stars ultimately to explode.

Table: 1.1: Characteristics of the Four Fundamental Forces

Force	Relative strength	Range	Importance
Strong	1	$10^{-13}$ cm	Holds nucleus together
Electromagnetic	$10^{-2}$	Infinite	Friction, tensions, etc.
Weak	$10^{-5}$	$10^{-13}$ cm	Nuclear decay
Gravitational	$10^{-39}$	Infinite	Organizes universe

Dr. Abdus Salam proposed a theory in 1967, according to which the weak and electromagnetic forces can be regarded as part of a single force, called the electroweak force. Other new theories, called grand unification theory, have been proposed which combine the strong and electroweak forces into single frame work.

## 1-2: Contact Forces

The fundamental forces of nature act at a distance; their effects can be experienced when the particles are not in contact. A second category is contact forces. These are forces that two objects exert on each other when they are physically in contact with each other. Contact forces are not fundamental forces, instead they arise fundamentally from electric forces acting in complicated ways.

The force a spring exerts on an object is an example of an electric force. Inside the spring are metal atoms that are bound together by electric forces. These electric forces keep the metal atoms a certain distance apart called the equilibrium distance. When you stretch a spring each atom is pulled a little bit out of the equilibrium distance. The electric forces try to pull the atoms back into the equilibrium position. The net result of all the electric forces on the atoms is what causes the end of the spring to pull on you, i.e. to exert a macroscopic force.

The empirical law for a spring is known as Hooke's law:

$$F = -k x \quad \dots(3)$$

The tension in a rope is another example of an electric force. When you pull on one end each atom electrically tugs on its neighbors, but unlike a spring, the rope doesn't stretch because it can't uncoil. The pull is transmitted to the other end of the rope, much like a chain link.

Whenever any object is pressing against another, there is a contact force between the two objects, known as the normal force. This force is a result of repulsion between the atoms of the two objects. The magnitude of the normal force depends on how hard the two objects press against each other. The direction of the normal force acting on an object, however, is always perpendicular to the surface.

Friction is one of the important example of an electric force. It is discussed in next section.

### 1-3: Frictional Forces

We classify as forces of friction those "passive" forces that tend always to prevent or retard motion. The force exerted by a surface on a body in contact with it can be resolved into two components, one perpendicular to the surface and one tangential to the surface. The perpendicular component is called the normal force and the tangential component is called friction.

Friction is of two main kinds: 1) Static friction and 2) kinetic or sliding friction. In static friction the two engaging surfaces do not rub against each other; their engagement somehow prevents relative motion. In kinetic friction, on the other hand, rubbing does occur, and accompanied by the generation of heat.

Static friction is useful in our daily life. It enables to walk, or to start, stop or change the direction of wheeled vehicles. The possibility of making string, rope and fabrics of all kinds depends on static friction, the strength and durability of the materials depending upon the friction between the fibers. Without static friction, it would not possible to tie knots! Also the action of belts, pulleys, and friction drives in machinery depends on static friction.

Kinetic friction is responsible for the dissipation of energy in machinery and its wearing out. The action of the clutch of an automobile during starting and the action of breaks is due to kinetic friction.

A third heading under which friction is sometimes classified is rolling friction, such as we have when a ball is rolled. The coefficient of rolling friction is very small in comparison with the other two coefficients. Because of the extreme smallness of rolling friction that ball and roller bearings are used in machinery.

#### Main Phenomenon:

Guillaume Amontons in 1699 found the empirical relationship that frictional forces from a surface do not exceed an amount proportional to the normal force exerted by the surface on the object,

$$f \leq \mu N \quad \dots(4)$$

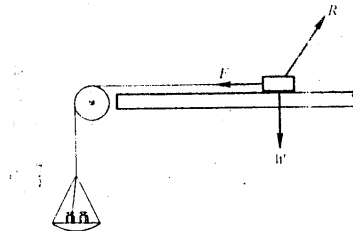
where  $\mu$  is the coefficient of friction.

The coefficient of friction depends on many variables, such as the nature of the materials, surface finish, surface films, temperature, and extent of contamination.

Suppose that you have a block at rest on a horizontal surface. By Newton's 2<sup>nd</sup> law, the force of friction is zero. Now suppose you apply a small measurable force  $F$  to it, and observe that the block does not move. By Newton's 2<sup>nd</sup> law, the force of static friction is equal in magnitude to  $F$ . Now suppose that you increase  $F$  and note that the block still

does not move. The force of static friction increases as well, being equal to  $-F$  always. If  $F$  is increased, there will be a definite value of  $F$  for which the block slips.

Fig. 1-1:  
System for investigating  
frictional phenomena.



The smallest force necessary to start motion is the maximum force of static friction  $\mu_s N$ . These observations can be summarized by the relation for the magnitude of the force of static friction  $f_s$ :

$$\text{Static friction : } f_s \leq \mu_s N \quad \text{.....(5)}$$

Where  $\mu_s$  is the coefficient of static friction, which depends on the two surfaces in contact and  $N$  is the normal force.

Once the block begins to move kinetic friction acts on the block. This frictional force is usually less than the static friction. Its empirical relationship is:

$$\text{Kinetic friction : } f_k = \mu_k N \quad \text{.....(6)}$$

Where  $\mu_k$  is the coefficient of kinetic friction and  $N$  is the normal force.

Table: 1.1- Coefficients of friction

Material	$\mu_s$	$\mu_k$
Steel on steel	0.78	0.42
Nickel on nickel	1.10	0.53
Wood on wood	0.35	0.15
Glass on glass	0.95	0.40
Rubber on dry concrete	1.00	0.80
Ice on ice	0.05	0.04

#### Laws of Friction:

Leonardo da Vinci (1452 – 1519) found that frictional force is proportional to load and is independent of the area of the engaging surfaces. These laws were rediscovered by Amontons in 1699 and verified by Coulomb in 1781, who was the first to distinguish clearly between static and kinetic friction. The work of these investigators has led to the formulation of the following laws of friction:

1. Limiting static friction is greater than kinetic friction.
2. Kinetic friction is independent of the speed of rubbing, if moderate.
3. Frictional force is proportional to the normal force between the surfaces.
4. Frictional force is independent of the area of the surfaces in contact.

Although these so-called laws of friction still commonly quoted quite uncritically, are useful for the purposes of the preliminary orientation in the subject for rough engineering applications, a little experimentation and reflection will show that they really cannot be accepted without considerable qualification.



### Modern Theory of Friction:

The true area of contact between the engaging surfaces is only a minute fraction of the apparent area. From this it follows that the pressure at the actual contacts must be enormous. Bowden's investigations have shown that it is, in fact, sufficient to cause even such a hard metal as steel to flow plastically.

### Microscopic basis:

Even though a highly polished object may appear smooth, when examined through a microscope it appears very rough, having many tiny surface irregularities, as shown in the fig. 1-2

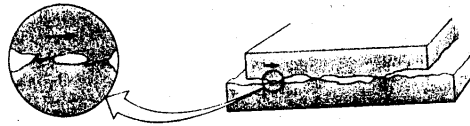


Fig. 1-2: Microscopic examination of a highly polished surface reveals irregularities.

When two objects are placed in contact, the many contact points resulting from the (microscopic) rough edges actually become welded together by electric forces. When one object moves across another, these tiny welds rupture and continually reform. The net result is friction – a force parallel to the surface which opposes the motion of the object. Since the number of welds is proportional to pressure from the object on the surface, the force of friction is proportional to the normal force on the object.

If the relative speed of the two surfaces is great enough, there may be local melting at certain contact areas even though the surface as a whole may feel only moderately warm. The *stick and slip* events are responsible for the noises that dry surfaces make when sliding across one another.

### Role of surface contamination:

The roughness of the surfaces has less role in determining the coefficient of friction. Of far greater importance is the state of cleanliness of the surfaces. According to Bowden and Tabor, even a single molecular layer of grease from the atmosphere or from the fingers may produce a considerable change in friction.

Under good vacuum conditions, and without special heat treatment of metal surfaces to remove contamination, frictional effects are observed to be much larger than in air. This must be due to the evaporation of a significant proportion of the adsorbed gas, ordinarily functioning as a species of lubricant.

### Effects of localized heating:

Highly localized frictional heating is occurred due to extreme smallness of the area of contact. More high temperature is obtained by connecting dissimilar metals sliding over one another, the regions in actual contact are quite commonly raised to temperatures of the order of 1000 °C.

These local hot spots play an important role in a number of processes involving rubbing, for example polishing, the frictional welding of plastics, and the initiation of chemical reactions brought about by friction. As an example calcite (m.p. 1333 °C) is not polished by cuprous-oxide (m.p. 1233 °C) but is readily polished by Zinc Oxide (m.p. 1850 °C).

**Lubrication:**

Lubrication prevents actual contact between surfaces and eliminates sliding friction. On the other hand it introduces a new item of resistance to movement of its own, the resistance due to the viscosity of the lubricant. Nevertheless, the overall effect of lubrication with a suitable oil or grease is to reduce resistance to motion.

**Rolling friction:**

Rolling resistance is almost unaffected by the presence of lubricants. Rolling friction arises primarily from losses associated with the failure of an elastically deformed body to regain its original configuration, immediately on release of the stress. This necessarily involves a loss of mechanical energy to heat.

**1-4: Uniform Circular Motion**

Knowing about motion in a constant direction but with uniformly varying speed (constant acceleration), let us now consider the converse of this – the motion of a particle having constant speed but moving in a uniformly varying direction, i.e., in a circular path. As the velocity of the particle (distinct from speed) is constantly changing, the particle must be accelerated.

Let's find the magnitude and direction of instantaneous acceleration in its circular path. When the time interval taken is made infinitesimally small, the mean and instantaneous accelerations become identical.

Consider a finite small time interval  $\delta t$  during which particle moves from point P to Q.

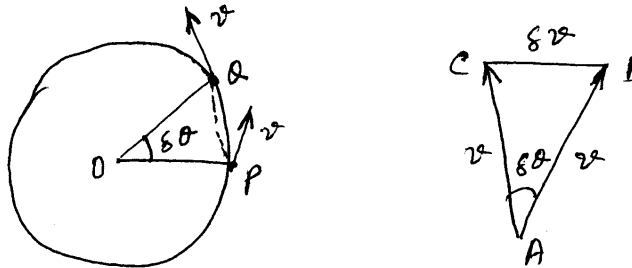


Fig. 1-3: Vector construction for derivation of centripetal acceleration.

Considering the fig., let AB & AC be drawn to represent the velocities at points P and Q respectively.  $v$  represent constant speed of the particle at points P and Q. AB represents the velocity at the beginning of the time interval considered and AC at the end. From the vector triangle  $\delta v$  is the change of velocity compounded with the initial velocity to produce final velocity.

Now for small angle

$$\text{Arc PQ} = \text{line PQ}$$

& triangles OPQ & ABC are similar

$$\therefore BC = AB \delta\theta$$

$$\text{or } \delta v = v \delta\theta$$

dividing both side by  $\delta t$ ,

$$a = \delta v / \delta t = v (\delta\theta / \delta t)$$

with  $\delta\theta$  infinitesimally small, BC becomes perpendicular to AB so that the direction of the acceleration is radial and toward the centre of the circle. This acceleration is known as centripetal acceleration. Because

$$\delta\theta = PQ / OP$$

$$\therefore \delta\theta / \delta t = \omega = (PQ / \delta t) / OP = v / r$$

so we obtain three alternative expressions for centripetal acceleration

$$a = v\omega = v^2 / r = \omega^2 r \quad \dots\dots(7)$$

$$\text{since } F = ma$$

$$\text{therefore } F = m v^2 / r$$

it is called centripetal force.

$\dots\dots(8)$

## 1-5: The Conical Pendulum

### Definitions:

**Spherical pendulum:** A heavy bob attached to a light inextensible rod or cord which is free to swing in any direction about a fixed point.

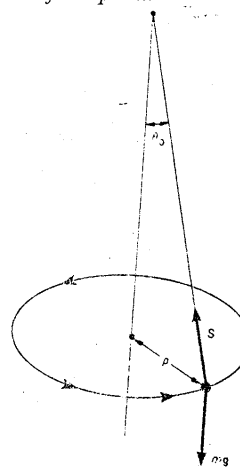
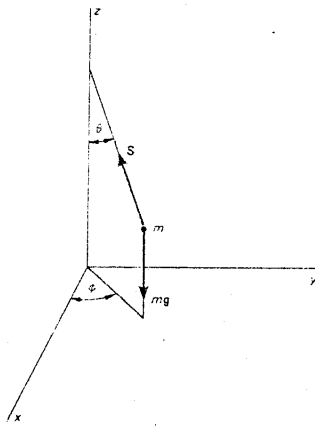


Fig. 1-4: The spherical pendulum. Fig. 1-5: The conical case of the spherical pendulum.

**Conical Pendulum:** It consists of a bob suspended from a point by a light string, which describes a circular orbit in a horizontal plane.

Conical pendulum is a special case of spherical pendulum where  $\theta = \theta_0 = \text{constant}$

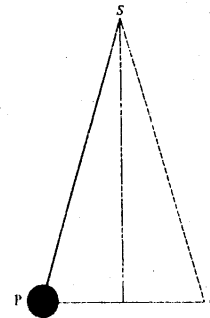
In fig. 1-6, the circle PQ described by the bob is seen edge on. If viewed in this way, the bob, which actually moves with constant speed around its circular orbit, appears merely to move back and forth in a straight line. Which is very direct and convenient way to observe simple harmonic motion, defined as a projection of uniform circular motion.

### Mathematical Analysis:

From fig. 1-6

$$\begin{aligned}\text{Mass of the body} &= m \\ \text{Weight of the body} &= mg \\ \text{Radius of circular path: } R &= L \sin \theta \\ \text{Tension in the string} &= T\end{aligned}$$

Resolving  $T$  into radial and vertical components, Fig. 1-6: Apparent motion of the bob of a conical pendulum when viewed in the plane of motion.



$$\begin{aligned}T_r &= -T \sin \theta \\ &\& T_z = T \cos \theta \\ \text{Form Newton's 2}^{\text{nd}} \text{ law} \quad F &= ma \\ \text{or} \quad \Sigma F &= T + mg = ma \quad \dots(9)\end{aligned}$$

$$\begin{aligned}\text{Since there is no vertical acceleration} \\ \therefore \quad \Sigma F_z = T_z - mg &= 0 \\ \text{or} \quad T \cos \theta &= mg \quad \dots(10)\end{aligned}$$

$$\begin{aligned}\text{And radial component is} \\ \Sigma F_r = T_r = m a_r \\ \text{or} \quad -T \sin \theta = -mv^2/R \quad [a_c = -v^2/R] \quad \dots(11)\end{aligned}$$

Dividing eq. (11) with eq. (10), we get

$$\begin{aligned}\frac{-T \sin \theta}{T \cos \theta} &= \frac{-mv^2/R}{mg} \\ \text{or} \quad g \tan \theta &= v^2/R \\ \text{or} \quad v^2 &= R g \tan \theta \\ \text{or} \quad v &= \sqrt{R g \tan \theta} \quad \dots(12)\end{aligned}$$

Eq. (12) gives constant speed of the pendulum.

Let, time for one complete revolution =  $t$

We have for one revolution

$$\begin{aligned}v &= R \omega \quad \text{or} \quad v = 2\pi R / t \quad [\text{angular velocity} = \omega = \theta / t] \\ \text{or} \quad t &= 2\pi R / v \quad \dots(13)\end{aligned}$$

From eqs. (12) & (13), we have

$$\begin{aligned}t &= 2\pi R / \sqrt{R g \tan \theta} \\ \text{or} \quad t &= 2\pi \sqrt{R / g \tan \theta}\end{aligned}$$

Since  $R = L \sin \theta$ , so

$$t = 2\pi \sqrt{\frac{L \sin \theta}{g \tan \theta}}$$

or  $t = 2\pi \sqrt{\frac{L \cos \theta}{g}}$  .....(14)

which gives time period of conical pendulum.

Eq. (14) shows that time period of the conical pendulum is independent of mass of the body.

## 1-6: The Rotor

The rotor consists of a hollow cylinder that rotates about a vertical axis with an angular speed  $\omega$ . With the person against the spinning wall of the rotor, the floor drops out, but the person remains "pinned" to the wall. It is an amusement park ride. What is the minimum steady angular velocity  $\omega$  which allows the floor to be dropped away safely?

Let

Radius of the drum =  $R$

Mass of the person =  $M$

Coefficient of friction

between drum & person =  $\mu$

Fig. (1-8) shows that the forces on the man are

1. weight,  $w$
2. frictional force,  $f$
3. normal force by the wall,  $N$

We have from Newton's 2<sup>nd</sup> law

$$F = ma$$

The force along z-direction is

$$\Sigma F_z = f_z - mg = ma_z \quad \text{.....(15)}$$

since there is no upward or downward motion, therefore

$$f_z - mg = 0$$

Also radial component of the force is,

$$\Sigma F_r = -N = ma_r = -mv^2/R \quad \text{..... (16)}$$

By the law of static friction

$$f_s \leq \mu_s N = \mu_s m v^2 / R$$

$$\text{and } f_s = mg$$

and we have

$$mg \leq \mu_s m v^2 / R$$

$$\text{or } v \geq \sqrt{gR / \mu_s} \quad \text{.....(17)}$$

The smallest value of  $v$  that will work is :

$$v_{\min} = \sqrt{gR / \mu_s} \quad \text{.....(18)}$$

Also

$$\omega_{\min} = \sqrt{g / \mu R} \quad [v = R\omega \quad \text{.....(19)}]$$

For cloth on wood  $\mu$  is at least 0.3, and if the drum has radius 6 ft, then

$$\omega_{\min} = [32 / (0.3 \times 6)]^{1/2} = 4 \text{ rad/s}$$

The drum must make at least  $\omega / 2\pi = 0.6$  turns per second.

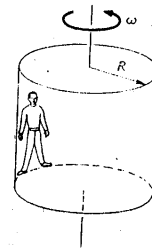


Fig. 1-7

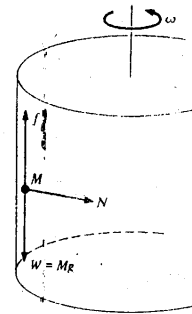


Fig. 1-8

## 1-7: Banked Curve

Bank means making the edge of the road elevated or having steep slope. At turnings we usually make one side of the road high for smooth speed of the vehicles.

When a car turns a corner it can be considered to be moving in the arc of a circle. The force required for the car to move in a

Circular path on a horizontal road is provided at the contact between the tyres and the road as a friction force. If the road surface at the corner was a sheet of ice and, thus, frictional forces were very low then the car would not turn the corner but continue with a constant velocity---keep on in the same straight line. The friction force has to be equal to  $mv^2/r$  if the car is to move the corner without skidding. Skidding is when the vehicle tends to continue in its straight line path.

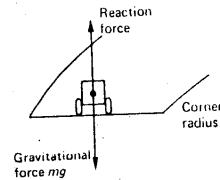


Fig. 1-9: Forces on a car moving round a corner with no banking.

In order not to rely solely on the friction forces to keep a car going round a corner banking can be used. In the figure, the road is horizontal and the entire cornering force must be provided by friction. There is no net force on the car in the vertical direction and, thus, the gravitational force acting on the car must be counter-balanced by a reaction force from the road. In this figure the corner is banked. The gravitational force acting on the car must be balanced by component of the reaction force, the reaction force being at right angles to the surface. Thus

$$R \cos \theta = mg$$

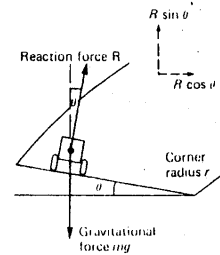


Fig. 1-10: Forces on car with banking.

The other component of the reaction force  $R \sin \theta$  is directed towards the center of the circle in which the car is moving and thus can provide the centripetal force.

$$R \sin \theta = mv^2 / r$$

The above equation assumes that the entire centripetal force is provided by the reaction force component and that there is no contribution from friction. Dividing the two equations gives

$$\frac{R \sin \theta}{R \cos \theta} = \frac{mv^2}{mg r}$$

$$\tan \theta = v^2 / rg$$

For the value of the velocity,  $v$ , given by this equation, cornering will occur even if there are no friction forces. There is a unique value of this velocity for a particular radius of curvature and angle of banking. Designers of roads usually design the angle and radius of the curve to suit the average speed of the vehicles.

## 1-8 : One-dimensional problems

In the usual mechanics problem the forces acting on a particle are specified, together with the initial values of the position and velocity of the particle. It is then desired to find the orbit traversed by the particle as a function of time.

From Newton's 2<sup>nd</sup> law, we have

$$F = ma \quad \dots(1)$$

This equation yields three simultaneous differential equations of motion of at least the second order, for which the solutions are not always available. We shall restrict ourselves to the simple problems, those for which exact solutions are available.

For simplicity in arriving at these equations of motion, we restrict ourselves to the motion of a particle along a straight line. If the component of the force along this line is a function of the time, the position, and the velocity of the particle, then the equation of motion for the particle has the form

$$F(x, v, t) = m \frac{d^2x}{dt^2} \quad \dots(2)$$

Where we have chosen the x-axis to lie along the line of motion. We will consider solutions of eq. (2) for various force functions.

### Definitions

**Constant force** : The force which do not depend upon time, velocity or position and will have constant acceleration.

**Non-constant force** : The force which depend upon position, time or velocity and will have variable acceleration.

**Analytical approach** : Solving the problems by using the methods of integration and differentiation.

**Numerical approach** : Solving the problems by using the numerical values of different functions such as, position and velocity.

## 1-9 : Force as a function of time

If the force is a function of time only, then the equation of motion is

$$F(t) = m \frac{d^2x}{dt^2} \quad \dots(3)$$

Integrating eq. (3), with respect to, from initial time to final time t,

$$\begin{aligned} m \frac{d^2x}{dt^2} &= F(t) \\ \int m \frac{d^2x}{dt^2} dt &= \int F(t) dt \\ m \left[ \frac{dx}{dt} \right]_{t_0}^t &= \int_{t_0}^t F(t) dt \\ m [v]_{t_0}^t &= \int_{t_0}^t F(t) dt \\ m(v - v_0) &= \int_{t_0}^t F(t) dt \end{aligned}$$

And

$$\begin{aligned} m \int \frac{dx}{dt} dt &= \int F(t) dt \\ m [x]_{t_0}^t &= \int_{t_0}^t F(t) dt \\ m \{v(t) - v(t_0)\} &= \int_{t_0}^t F(t) dt \\ \text{or } v(t) - v(t_0) &= (1/m) \int_{t_0}^t F(t) dt \\ \text{or } x(t) - x(t_0) &= (1/m) \int_{t_0}^t v(t) dt \\ \frac{dx}{dt}(t) - \frac{dx}{dt}(t_0) &= (1/m) \int_{t_0}^t F(t) dt \\ \text{or } \frac{dx}{dt}(t) &= \frac{dx}{dt}(t_0) + (1/m) \int_{t_0}^t F(t) dt \end{aligned} \quad \dots(4)$$

Integrating once again

$$\int dx(t) dt = \int dx(t) dt + \int (1/m) dt \int F(t) dt$$

$$\text{or } x(t) - x(t_0) = v_0 t - v_0 t_0 + 1/m \int dt \int F(t) dt \quad \dots(5)$$

If the necessary integrations in eq. (4) can be performed, then this equation gives an exact solution for the position of the particle as a function of time. If the integration cannot be performed, then we may get a numerical integration and obtain a numerical solution to the problem.

### Examples

1. A particle sliding down an incline. Here force as a function of time is constant force.

From eq. (4), we have

$$v = v_0 + a(t - t_0) \quad \dots(6)$$

where  $a = F/m$ , is the constant acceleration of the particle.

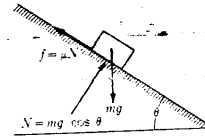


Fig. 1-11: Particle sliding down an incline.

2. Consider the problem of a particle which at time  $t = 0$  has the position  $x_0$ , the velocity  $v_0$  and is being acted on by the sinusoidal force.

$$F = F_0 \sin \omega t$$

The equation of motion is

$$X = (F_0 / x) \sin \omega t$$

Integrating

$$X = v_0 + F_0 / m\omega - F_0 / m\omega \cos \omega t$$

Integrating once again

$$X = x_0 + (v_0 + F_0 / m\omega) + F_0 / m\omega^2 \sin \omega t$$

This problem is of interest in connection with the scattering of electromagnetic radiation by free electrons. The scattering of an electromagnetic wave is produced by the electrons absorbing energy from the incoming electromagnetic wave and re-radiating this energy in all directions..

### 1-10 : Force as a function of velocity

Here we have

$$F(v) = m x \quad \dots(1)$$

Applying two ways to get the solution

$$\text{i) } F(v) = m dv / dt \quad \dots(2)$$

$$\text{ii) } F(v) = m v dv / dx \quad [dv / dt = (dv/dx) (dx/dt)] \quad \dots(3)$$

eqs. (2) & (3) after integration yields

$$t - t_0 = \int m dv / F(v) \quad \dots(4)$$

$$x - x_0 = \int m v dv / F(v) \quad \dots(5)$$

eqs. (4) & (5) are solutions of eq. (1).

#### Examples :

1. A particle moving in a resistive medium.
2. Projectile motion of a particle under the action of the force of gravity and the damping produced by the air through which the particle must move.



### 1-11 : Force as a function of position

The one-dimensional equation of motion for the rectilinear motion of a particle under the action of a force that is a function of position only is

$$F(x) = m \, dv/dt \quad \text{.....(1)}$$

Integrating w.r.t.  $x$ , we have

$$\int F(x) \, dx = \int m \, (dv/dt) \, dx \quad \text{.....(2)}$$

$$\text{since Work} = \int F(x) \, dx \quad \text{.....(3)}$$

lets define a potential energy as a function  $U(x)$ ,

$$F(x) = - (d/dx) U(x)$$

Integrating

$$\int F(x) \, dx = \int dU(x)$$

$$\int F(x) \, dx = - \int U(x) = U(x_0) - U(x) \quad \text{.....(4)}$$

from eqs. (3) & (4)

$$\text{Work} = U(x_0) - U(x) \quad \text{.....(5)}$$

Eq. (5) shows that the work performed by the force  $F(x)$  is equal to the difference between the initial and final values of potential energy of the particle.

From right hand side of eq. (2), after changing limits from  $x$  to  $t$  and letting  $dx = v \, dt$ , we have

$$\begin{aligned} \int m \, (dx/dt) \, dx &= \int m \, (dv/dt) \, v \, dt = \int d/dt \left( \frac{1}{2} m v^2 \right) dt \\ &= \left[ \frac{1}{2} m v^2 \right] = \frac{1}{2} m v^2 - \frac{1}{2} m v_0^2 \end{aligned} \quad \text{.....(6)}$$

put  $\frac{1}{2} m v^2 = T$ , as the kinetic energy

$$\int m \, (dv/dt) \, dx = T(x) - T(x_0) \quad \text{.....(7)}$$

From eqs. (2), (3), (5) & (7), we have

$$W = \int F(x) \, dx = U(x_0) - U(x) = T(x) - T(x_0)$$

$$\text{Or } U(x) - T(x) = U(x_0) - T(x_0) \quad \text{.....(8)}$$

Eq. (8) expresses the principle of conservation of total energy,  $E$

$$E = T(x) + U(x) \quad \text{.....(9)}$$

Eq. (8) show that the work done on a particle by a force is equal to the change in its kinetic energy. It is applicable in electrical and gravitational forces. The example of a simple harmonic oscillator illustrates this principle.

### 1-12 : Drag Forces

Consider the motion of falling objects with

Air present. In figure we show two forces.

One is the weight  $mg$ . The other is the

contact force that is exerted by the air;

Shown as  $f_{\text{air}}$

From Newton's 2<sup>nd</sup> law of motion

$$\Sigma F = ma$$

$$\text{or } mg - f_{\text{air}} = ma \quad \text{.....(1)}$$

We want to establish a formula that gives the dependence of  $f_{\text{air}}$  on the speed.

One way is to determine experimentally how the acceleration  $a$  varies with the speed  $v$ , and to fit the data with a formula for  $a$  as a function of  $v$ . Then we substitute this expression for  $a$  into eq. (1) and solve for  $f_{\text{air}}$ . We will not use this method here because it is difficult to avoid errors in taking the slopes in obtaining the acceleration.

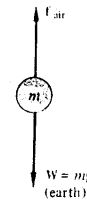


Fig. 1-12: The force diagram for a sphere falling in air.

Lets reverse the above logic. We will assume an expression for  $f_{\text{air}}$  and use it to solve for  $y(t)$ . If the calculated  $y(t)$  agrees with the experimental one, then the expression for  $f_{\text{air}}$  is correct.

Assume two possible formulas for  $f_{\text{air}}$  :

$$f_{\text{air}} = b v^2 \quad \text{.....(2)}$$

$$f_{\text{air}} = c v \quad \text{.....(3)}$$

From the figure, the velocity of the ball increase until it reaches a constant value called the terminal velocity ,  $v_t$  .

While moving at this constant terminal velocity, the acceleration is zero. So from eq. (1)

$$f_{\text{air}} = mg, \quad \text{at } v = v_t \quad \text{.....(4)}$$

from eqs. (2) & (4), we have

$$mg = b v_t^2$$

$$\text{or} \quad b/m = g / v_t^2 \quad \text{.....(5)}$$

Also from eqs. (3) & (4), we have

$$mg = c v_t$$

$$\text{or} \quad c/m = g / v_t \quad \text{.....(6)}$$

In eqs. (5) & (6) we can determine  $b/m$  &  $c/m$  by measuring the terminal velocity.

Now we can predict the acceleration of the sphere from eq. (5)

$$b = mg / v_t^2$$

From eqs. (2) & (5)

$$f_{\text{air}} = mg v^2 / v_t^2 \quad \text{.....(7)}$$

from eqs. (1) & (7)

$$mg - mg v^2 / v_t^2 = ma$$

$$\Rightarrow a = g (1 - v^2 / v_t^2) \quad \text{.....(8)}$$

similarly from eqs. (1), (3) & (6), we get

$$a = g (1 - v / v_t) \quad \text{.....(9)}$$

If we will check with experimental values, we see that eq. (2) gives better values for  $f_{\text{air}}$  .

We can write eq. (2) in more general form for the mechanics for fluids.

$$F_{\text{air}} = C_D (1/2 \rho A) v^2 \quad \text{.....(10)}$$

Where

$\rho$  is mass per unit volume of the fluid

$A$  is the cross-sectional area of the sphere

$C_D$  is called drag coefficient which has a

Value 0.4 for many velocities

From eq. (10) we can calculate the drag force on spheres of different sizes that are moving in different fluids.

For example eq. (10) can be used for a sphere with a diameter of 1cm falling in air at speeds between 0.6 m/s and 300 m/s. For very small speeds, the drag force is found to be proportional to the speed  $v$  instead of  $v^2$  and so eq. (3) is good for low speeds.

### 1-13 : Projectile with air resistance

*Any object that is given an initial velocity and that subsequently follows a path determined by the gravitational force acting on it and by the frictional resistance of the atmosphere is called a projectile.*

If you have observed a cricket ball in motion ( or , for that matter, any object thrown in air ), have observed projectile motion. For an arbitrary direction of initial velocity, the ball moves in a curved path. The path followed by a projectile is called its trajectory .

The examples are—a missile shot from a gun, a rocket after its fuel is exhausted, an object dropped from an airplane. Consider a particle as a projectile which the air exerts a force  $R$  acting in a direction opposite to the velocity. Resolving forces and acceleration in order to obtain equations of motion in scalar form, along horizontal and vertical directions,

$$m a_x = -R \cos \theta \quad \dots(1)$$

$$\& m a_y = -R \sin \theta - mg \quad \dots(2)$$

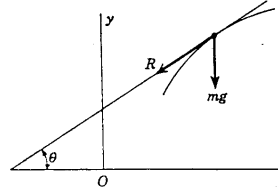


Fig. 1-13: The forces acting on a projectile

The resistance experienced by a given projectile depends on its speed and on the density of the air. Regarding the air as stratified into horizontal layers each of constant density, so that the density is a function of  $y$  only, we may express  $R$  in the form

$$R = R(y, v) \quad \dots(3)$$

Since

$$\cos \theta = x/v \quad \dots(4)$$

$$\& \sin \theta = y/v \quad \dots(5)$$

$$m a_x = -R x/v = R(y, v) x/v$$

from eqs. (1) & (2)

$$a_x = -\Phi x \quad \& \quad a_y = -\Phi y - g \quad \dots(6)$$

where  $\Phi$  is a function of  $y, v$ , namely

$$\Phi(y, v) = R(y, v) / mv \quad \dots(7)$$

The mathematical problem of the determination of the trajectory is made much more difficult by the fact that there is no physically valid formula expressing  $R$  as a function of  $v$ . For small values of  $v$ ,  $R$  varies as  $v$ , but this simple law breaks down before we reach those velocities which are of interest in ballistics.

To make complicated physical problem to a simple one, we shall confine our attention below to the case where  $R$  is independent of  $y$  and devote particular attention to the cases where  $R$  varies as  $v^2$  or as  $v$ .

### Resistance Independent of Height

In projectile motion (figure above), resolving forces and acceleration along the tangent and normal to the trajectory,

$$mv dv/ds = -R - mg \sin \theta \quad \dots(8)$$

$$\& m v^2/\rho = mg \cos \theta \quad \dots(9)$$

where  $\rho$  is the radius of curvature

Now if the resistance depends on the speed only, so

$$R = R(v)$$

Lets write

$$R = mg \phi(v)$$

We have

$$\rho = -ds/d\theta$$

and eliminating  $ds$  from eqs. (8) & (9)

$$\frac{1}{v} \frac{dv}{d\theta} = \frac{\phi(v)}{\cos \theta} + \sin \theta$$

This is called the equation of the hodograph, since  $v, \theta$  are the polar coordinates of a point on the hodograph.

### Resistance varying as the square of the velocity

Let us take the law of resistance to be

$$R = mg \phi(v), \quad \phi(v) = C v^2 \quad \dots(10)$$

Where  $C$  is a constant.

### Resistance varying directly as the velocity

Although the law of resistance

$$R = mg Cv \quad \dots(11)$$

Is not accurate physically, it is so simple to treat mathematically that it is a useful approximation.

From eq. (7), we note that  $\Phi$  is now a constant ( $\Phi = gC$ ), and the eqs. (6) are easy to handle, because the var variable  $x$  and  $y$  are separated. We obtain on integration

$$x = x_0 + \frac{U_x}{\Phi} (1 - e^{-\Phi t}) \quad \dots(12)$$

$$\& y = y_0 - \frac{g}{\Phi} t + \left( \frac{U_y}{\Phi} + \frac{g}{\Phi^2} \right) (1 - e^{-\Phi t}) \quad \dots(13)$$

where  $x_0, y_0$  are the coordinates and  $U_x, U_y$  the components of velocity for  $t = 0$ .

When  $\Phi$  is small, these equations yield approximately

$$x = x_0 + U_x t - \frac{1}{2} \Phi U_x t^2 \quad \dots(14)$$

$$\& y = y_0 + U_y t - \frac{1}{2} g t^2 - \frac{1}{2} \Phi U_y t^2 (1 - 1/3 g t / U_y) \quad \dots(15)$$

in which the terms independent of  $\Phi$  correspond to the parabolic trajectory.

### **1-14 : Pseudoforces**

Consider relationship between two persons, Aslam and Shahid, who use different coordinate systems. Let us suppose that the positions of a particle as measured by Aslam are  $x$  and by Shahid are  $x'$ ; then the laws are as follows:

$$x = x' + s, \quad y = y', \quad z = z' \quad \dots(1)$$

where  $s$  is the displacement of Shahid's system relative to Aslam's. If we suppose that the laws of motion are correct for Aslam, how do they look for Shahid? From Eq. (1)

$$dx/dt = dx'/dt + ds/dt \quad \text{or} \quad d^2x/dt^2 = d^2x'/dt^2 + d^2s/dt^2 \quad \dots(2)$$

$$\text{or} \quad m d^2x/dt^2 = m d^2x'/dt^2 + m d^2s/dt^2 \quad \dots(3)$$

$$\text{or} \quad F_x = d^2x'/dt^2 + d^2s/dt^2 \quad \dots(4)$$

For non-accelerated frame of reference, when  $s$  is constant, then  $ds/dt = 0$ ,

Therefore the laws of Physics are same in both the systems.

If  $s = vt$ , where  $v$  is uniform velocity in a straight line,

Then  $ds/dt$  is not zero, but  $v$  is constant,

$$\text{so} \quad dv/dt = d^2s/dt^2 = 0$$

and the acceleration from eq.(2) is  $d^2x/dt^2 = d^2x'/dt^2$  ..... (5)

Now for accelerated frame of reference, from  $s = v_i t + \frac{1}{2} a t^2$  [for  $v_i = 0$ ]

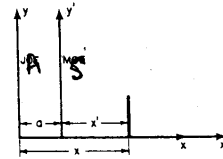
$$\text{Then} \quad ds/dt = a t \quad \text{and} \quad d^2s/dt^2 = a \quad \dots(6)$$

This means that although the laws of force from point of view of Aslam would look like

$$F_x = m d^2x/dt^2$$

the laws of force as looked upon by Shahid would appear as, from eqs. (4) & (6)

$$F_x = d^2x'/dt^2 + m a$$



That is, since Shahid's coordinate system is accelerating with respect to Aslam, the extra term  $ma$  comes in, and Shahid will have to correct his forces by that amount in order to get Newton's laws to work.

Here is an apparent, mysterious new force of unknown origin which arises, of course, because Shahid has the wrong coordinate system. This is an example of pseudo force, in an accelerated frame of reference, which is non-inertial frame of reference. (Appendix A gives something about inertial and non-inertial frames of reference). Other examples occur in coordinate systems that are rotating.

## 1-15 : Examples of Pseudo forces

### 1. Linear accelerator

A small sphere of mass  $m$  is hung from the ceiling of an accelerating rail car. According to the inertial observer at rest, the forces on the sphere are the tension  $T$  and the weight  $mg$ . The inertial observer concludes that the acceleration of the sphere of mass  $m$  is the same as that of the rail car and that this acceleration is provided by the horizontal component of  $T$ . Also, the vertical component of  $T$  balances the weight. Therefore, the inertial write Newton's 2<sup>nd</sup> law as

$$T + mg = ma$$

Which in component form becomes

For inertial observer

$$\Sigma F_x = T \sin \theta = ma \quad \dots(1)$$

$$\Sigma F_y = T \cos \theta - mg = 0 \quad \dots(2)$$

from eqs. (1) & (2), the inertial observer would calculate

$$a = g \tan \theta \quad \dots(3)$$

Therefore, since the deflection of the string from the vertical serves as a measure of the acceleration of the car, a simple pendulum can be used as an accelerometer.

According to the noninertial observer riding in the car, described in fig. b, the sphere is at rest and the acceleration is zero. Therefore, the noninertial observer introduces a pseudo (or fictitious) force,  $-ma$ , to balance the horizontal component of  $T$  and claims that the net force on the sphere is zero. In this noninertial frame of reference, Newton's 2<sup>nd</sup> law in component form gives, for non-inertial observer

$$\Sigma F'_x = T \sin \theta = ma \quad \dots(4)$$

$$\Sigma F'_y = T \cos \theta - mg = 0 \quad \dots(5)$$

These expressions are equivalent to (1) & (2); therefore the non-inertial observer gets the same mathematical results as the inertial observer. However, the physical interpretation of the deflection of the string differs in the two frames of reference.

### 2. Rotating system

An observer in a rotating system is another example of a non-inertial observer. Suppose a block of mass  $m$  lying on a horizontal frictionless turntable is connected to a string as shown in the figure. According to an inertial observer, if the block rotates uniformly, it

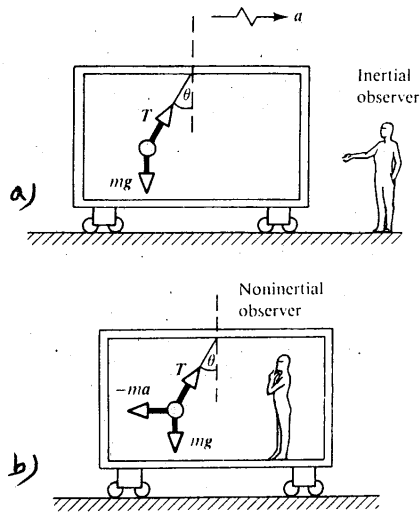


fig. 1-15: a) Inertial observer at rest outside the car.  
b) A non-inertial observer riding in the car.

undergoes a centripetal acceleration  $v^2 / r$ , where  $v$  is its tangential speed. The inertial observer concludes that this centripetal acceleration is provided by the force of tension in the string,  $T$ , and writes Newton's 2<sup>nd</sup> law  $T = mv^2 / r$ .

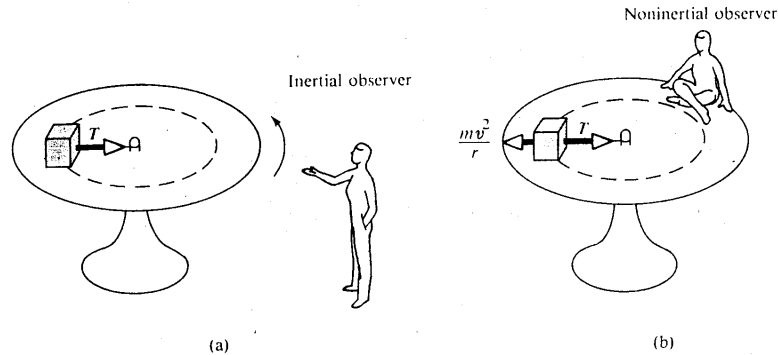


Fig. 1-16: a) According to inertial observer, the centripetal force is provided by the tension  $T$ .  
b) According to non-inertial observer, the block is not accelerating, so introduces a fictitious centrifugal force

According to a non-inertial observer attached to the turntable, the block is at rest. Therefore in applying Newton's 2<sup>nd</sup> law, this observer introduces a pseudo (or fictitious) outward force called the centrifugal force, of magnitude  $mv^2/r$ . According to the non-inertial observer, this centrifugal force balances the force of tension and therefore

$$T - mv^2/r = 0$$

You should be careful when using pseudo forces to describe physical phenomena. Remember that pseudo forces, such as centrifugal force, are used only in non-inertial, or accelerated frames of reference. When solving problems, it is generally best to use an inertial frame.

### 3. Pushing a jar of water

Lets push a jar of water along a table, with some acceleration. Gravity will act downward and there is also a pseudo force acting horizontally and in a direction opposite to the acceleration. The resultant of gravity and pseudo force makes an angle with the vertical, and during the acceleration the surface of the water will be perpendicular to the resultant force, i.e. inclined at an angle with the table, with the water standing higher in the rear side of the jar. When the push on the jar stops and the jar decelerates because of friction, the pseudo force is reversed, and the water stands higher in the forward side of the jar.

### 1-16 : Dynamics of a particle in a Rotating Coordinate System

Since the primary coordinate system is assumed to be an inertial system, then the fundamental equation of motion is

$$F = m d^2 r / dt^2$$

Writing the equation of motion in terms of the moving coordinates,

$$F - m A_0 - 2m\omega \times r - m\omega \times r - m\omega \times (\omega \times r) = m r \quad \dots(1)$$

The terms have been transposed in order to display them in the form of inertial forces to be added to the physical force  $F$ . The inertial terms have been given name.

The Coriolis force:

$$F_{Cor} = - 2m\omega \times r$$

The transverse force:

$$F_{trans} = - m\omega \times r$$

The centrifugal force:

$$F_{cent} = - m\omega \times (\omega \times r)$$

The remaining force ,  $- m A_0$  , is the inertial term due to transition of the coordinate system.

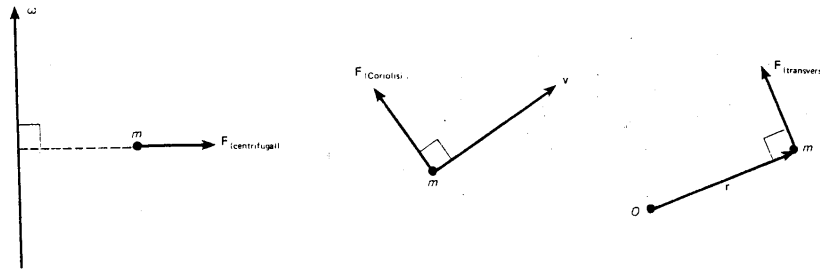


Fig. 1-17: Inertial forces (drawn separately) arising from rotating of the coordinate system.

Writing the equation of motion in the moving system as

$$F = m r$$

In which the total force is given by

$$F = F + F_{Cor} + F_{trans} + F_{cent} - m A_0 \quad \dots(2)$$

The four inertial terms on the right hand side all depend on the particular coordinate system in which the motion is described. They arise from the inertial properties of matter rather than from the presence of other bodies.

The Coriolis force is particularly interesting. It is present only if a particle is moving in a rotating coordinate system. Its direction is always perpendicular to the velocity vector of the particle in the moving system. The Coriolis force thus seems to deflect a moving particle at right angles to its direction of motion. This force is important, for example, in computing the trajectory of a projectile. Coriolis effects are also responsible for the circulation of air around high- or low-pressure areas on the earth's surface. Thus in the case of a high-pressure area the air tends to flow outward and to the right in the northern

hemisphere, so that the circulation is clockwise. In the southern hemisphere the reverse is true.

The transverse force is present only if there is an angular acceleration of the rotating coordinate system. This force is always perpendicular to the radius vector  $\mathbf{r}$ , hence the name transverse.

Finally, the centrifugal force is the familiar force arising from rotation about an axis. This force is always directed outward away from the axis of rotation and is perpendicular to that axis. If  $\theta$  is the angle between the radius vector  $\mathbf{r}$  and the rotation vector  $\omega$ , then the magnitude of the centrifugal force is clearly  $m\omega^2 \sin \theta$  or  $m\rho\omega^2$  where  $\rho$  is the perpendicular distance from the moving particle to the axis of rotation. The various forces are illustrated in the figure.

### EXAMPLE:

A bug crawls outward with constant speed  $v$  along the spoke of a wheel which is rotating with constant angular velocity  $\omega$  about a vertical axis. Find all the forces acting on the bug. First, let us choose a coordinate system fixed on the wheel, and let the x-axis point along the spoke in question. Then we have

$$\mathbf{r} = \mathbf{i}x = \mathbf{i}vt$$

$$\dot{\mathbf{r}} = \mathbf{i}v = \mathbf{i}v$$

$$\ddot{\mathbf{r}} = \mathbf{0}$$

for the equations of motion of the bug as described in the rotating system. If we choose the z-axis to be vertical, then

$$\omega = k\omega$$

The various forces are then given by the following:

Coriolis force

$$-2m\omega \times \dot{\mathbf{r}} = -2m\omega v(k \times \mathbf{i}) = -2m\omega v\mathbf{j}$$

Transverse force

$$-m\omega \times \mathbf{r} = 0 \quad (\omega = \text{constant})$$

Centrifugal force

$$\begin{aligned} -m\omega \times (\omega \times \mathbf{r}) &= -m\omega^2[k \times (k \times \mathbf{i}x)] \\ &= -m\omega^2(k \times \mathbf{j}x) \\ &= m\omega^2 x\mathbf{i} \end{aligned}$$

Thus, eq. (1) reads

$$\mathbf{F} - 2m\omega v\mathbf{j} + m\omega^2 x\mathbf{i} = 0$$

Here  $\mathbf{F}$  is the real force exerted on the bug by the spoke. The forces are shown in the figure.

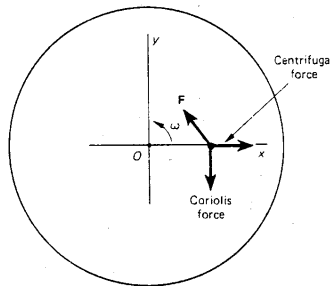


Fig. 1-18: Forces on a bug crawling outward along a radial line of a rotating turntable.



## 1-17 : Coriolis Force

Consider the turn-table experiment. Consider the body and the arms separately, from the point of view of the man who is rotating. After the weights are pulled in, the whole object is spinning faster, but observe, the central part of the body is not changed, yet it is turning faster after the event than before. So, if we were to draw a circle around the inner body, and consider only the objects inside the circle, their angular momentum would change; they are going faster. Therefore there must be a torque exerted on the body while we pull in our arms. No torque can be exerted by the centrifugal force, because that is radial. So that means that among the forces that are developed in a rotating system, centrifugal force is not the entire story, there is another force. This other force is called Coriolis force, and it has the very strange property that when we move something in a rotating system, it seemed to be pushed sidewise. Like the centrifugal force, it is an apparent force. But if we live in a system that is rotating, and move something radially, we find that we must also push it sidewise to move it radially. This sidewise push which we have to exert is what turned our body around.

Now let us develop a formula to show how this Coriolis force really works. Suppose Aslam is sitting on a rotating swing that appears to him to be stationary. But from the point of view of Shahid, who is standing on the ground and who knows the right laws of mechanics, the rotating swing is going around. Suppose that we have drawn a radial line on the rotating swing, and that Aslam is moving some mass radially along this line. We would like to demonstrate that a sidewise force is required to do that. We can do this by paying attention to the angular momentum of the mass. It is always going around with the same angular velocity  $\omega$ , so that the angular momentum is

$$L = mv_{\text{tang}} r = m\omega r \cdot r = m\omega r^2$$

So when the mass is close to the center, it has relatively little angular momentum, but if we move it to a new position farther out, if we increase  $r$ ,  $m$  has more angular momentum, so a torque must be exerted in order to move it along the radius. (If you walk along the radius in the rotating swing, you have to lean over and push sidewise.) The torque that is required is the rate of change of  $L$  with time as  $m$  moves along the radius.

If  $m$  moves only along the radius,  $\omega$  stays constant, so that the torque is

$$\tau = F_c r = dL/dt = d(m\omega r^2)/dt = 2m\omega r dr/dt$$

where  $F_c$  is the Coriolis force. What we really want to know is what sidewise force has to be exerted by Aslam in order to move  $m$  out at speed  $v_r = dr/dt$ .

This is  $F_c = \tau/r = 2m\omega v_r$ .

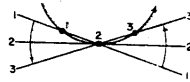


Fig. 1-19: Three successive views of a point moving radially on a rotating turntable.

Now that we have a formula for the Coriolis force, let us look at the situation a little more carefully, to see whether we can understand the origin of this force from a more elementary point of view. We note that the Coriolis force is the same at every radius, and is evidently present even at the origin. But it is especially easy to understand it at the origin, just by looking at what happens from the inertial system of Shahid, who is standing on the ground. Figure shows three successive views of  $m$  just as it passes the Origin at  $t = 0$ . Because of the rotation of the rotating swing, we see that  $m$  does not move in a straight line, but in a curved path tangent to a diameter of the rotating swing where  $r = 0$ . In order for  $m$  to go in a curve, there must be a force to accelerate it in the absolute space. This is the Coriolis force.

This is not the only case in which the Coriolis force occurs. We can also show that if an object is moving with constant speed around the circumference of a circle, there is also a Coriolis force. Aslam sees a velocity  $v_a$  around the circle. On the other hand, Shahid sees  $m$  going around the circle with the velocity  $v_j = v_a + \omega r$ , because  $m$  is also carried by the rotating swing. Therefore we know that the force really is, namely, the total centripetal force due to the velocity  $v_s$ , or  $m v_j^2 / r$ ; that is the actual force. Now from Aslam's point of view, this centripetal force has three pieces. We may write it all out as follow:

$$F_r = m v_s^2 / r = - m v_a^2 / r - 2m v_a \omega - m \omega^2 r$$

Now,  $F_r$  is the force that Aslam would see. Let us try to understand it. Would Aslam appreciate the first term? Yes! he would say, "even if I were not turning, there would be a centripetal force if I were to run around a circle with velocity  $v_a$ ." This is simply the centripetal force that Aslam would expect, having nothing to do with rotation. In addition, Aslam is quite aware that there is another centripetal force that would act even on objects which are standing still on his rotating swing. This is the third term. But there is an other term in addition to these, namely the second term, which is again  $2m\omega v$ . This Coriolis force  $F_c$  was tangential when the velocity was radial, and now it is radial when the velocity is tangential. In fact, one expression has a minus sign relative to the other. The force is always in the same direction, relative to the velocity, no matter in which direction the velocity is. The force is at right angles to the velocity, and of magnitude  $2m\omega v$ .

Chapter 2:

## WORK & ENERGY

## Chapter 2

### 2-1 : Work and Energy

Conservation laws assert that some quantity is conserved, which means that some quantity remains constant even when matter suffers drastic changes involving motions, collisions, and reactions. We apply these conservation laws to make predictions about some aspects of the motion of a particle or of a system of particles when it is undesirable or impossible to calculate the full details of the motion from Newton's 2<sup>nd</sup> law. Law of conservation of energy is one of the most fundamental laws of nature. It remains valid even when we step outside of the realm of Newtonian Physics and enter the realm of relativistic physics or atomic physics, where Newton's law fail.

Energy and work are closely related. We will see that the work done by a force on a body is related to the change of the kinetic energy of the body.

### 2-2 : Work

The word "work" means different things to different people. In Physics, the technical meaning is not the same as the common meaning. Taking the simple case of motion along a straight line, we define the work done by the force  $F_x$  on the particle as it moves some given distance  $\Delta x$  is defined as the product of the force and the displacement,

$$W = F_x \Delta x \quad \dots(1)$$

Eq. (1) gives the work done by a single force. If several forces act, then we add the amount of work done by all the forces acting on the particle, we obtain the net amount of work done by all these forces together,

$$W = F_{\text{net},x} \Delta x \quad \dots(2)$$

If the motion of the particle and the force are not along the same line, then we must generalize eq. (1). Consider a particle moving along some arbitrary curved path, with a constant force. The work done by this constant force during a displacement  $s$  is

$$W = F s \cos \theta \quad \dots(3)$$

Many forces are not constant. So if the value of the force is different for different positions, then the work done on a particle by such varying force is defined as the product of the average value of the force and the displacement,

$$W = \langle F_x \rangle \Delta x \quad \dots(4)$$

### 2-3 : Negative and Positive Work

Work is not a vector quantity. It is a scalar. Even though force and displacement are vectors, we do not give direction of their product. Work done by a force can be either positive or negative.

If a force  $\mathbf{F}$  doing work on an object and the displacement is in the opposite direction of the force, then

$$W = F s \cos \theta = F s \cos 180^\circ = F s (-1) = -F s$$

So we see that the force does negative work on an object. Positive work and negative work have very different effects. For example, when you push a bicycle in the direction of its motion, you speed the bicycle up. The work done is positive. But when you push on a bicycle in a direction opposite to its motion, you slow the bicycle down to a stop. Negative work tends to stop the motion.

Consider a force is lifting an object. If the object is not accelerating appreciably, then the value of the force is  $mg$ , the weight of the object. The work done will be

$$W = F s \cos \theta = (mg)(h)(1) = mgh$$

Now if a force is lowering the same object, the supporting force does negative work,

$$W = F s \cos \theta = (mg)(h)(-1) = -mgh$$

The work done by a supporting force balances the pull of gravity, we call this work as work done against gravity.

## 2-4 : Work done in One Dimension

### a) Constant force case

If the net force is constant, then from Newton's 2<sup>nd</sup> law,

$$a = \Sigma F / m \quad \text{.....(1)}$$

To determine the velocity after it has moved  $\Delta x$ , we can use the following formula

$$v_f^2 = v_i^2 + 2a\Delta x \quad \text{.....(2)}$$

Rearranging by substituting  $a = F_o / m$  and multiplying by  $(m / 2)$ ,

$$F_o \Delta x = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \quad \text{.....(3)}$$

Put  $F_o \Delta x = W$ ,

$$W = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \quad \text{.....(4)}$$

$$\text{Or } W = KE_f - KE_i = \Delta KE \quad \text{.....(5)}$$

Since kinetic energy of the particle is :  $K = \frac{1}{2} m v^2$ , therefore eq. (5) shows that the work done by the sum of the forces on an object equals the change in its kinetic energy.

### b) Variable force (in one dimension)

We have from Section 2-2,

$$W = F_{\text{net},x} \Delta x \quad \text{.....(1)}$$

If we have variable net force  $\Sigma F$ , which is directed along the line of motion as the object moves on the x-axis from  $x = x_1$  to  $x = x_2$ . Then generalizing eq.(1) in integral form.

$$W = \int_{x_1}^{x_2} \Sigma F_x dx \quad \text{.....(2)}$$

The integral in eq. (2) for the force graphed in the figure is equal to the area shown shaded in this figure.

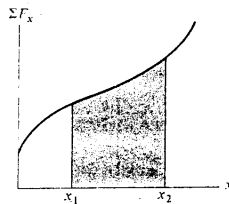


Fig. 2-1: *The work of the net force as the object moves from  $x = x_1$  to  $x = x_2$  is equal to the shaded area.*

## 2-5 : Work done in 2 dimensions

Consider an object moving in the x-y plane, the work done on it by a particular force  $\mathbf{F}$  that is constant. We allow the possibility of additional forces. We identify two points, A specified by  $(x_A, y_A)$  and B specified by  $(x_B, y_B)$ , and ask how much work is done by  $\mathbf{F}$  in displacing the object from A to B. We divide the path into a large number  $N$  of small displacements,  $\Delta \mathbf{s}_n = \Delta x_n \mathbf{i} + \Delta y_n \mathbf{j}$ , Because  $\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j}$  is constant, we have

$$\begin{aligned} W &= \sum \mathbf{F} \cdot \Delta \mathbf{s}_n = \sum (F_x \mathbf{i} + F_y \mathbf{j}) \cdot (\Delta x_n \mathbf{i} + \Delta y_n \mathbf{j}) \\ &= \sum F_x \Delta x_n + \sum F_y \Delta y_n \\ &= F_x \sum \Delta x_n + F_y \sum \Delta y_n \end{aligned} \quad [\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z]$$

$$\text{or} \quad W = F_x (x_B - x_A) + F_y (y_B - y_A) \quad \dots (1)$$

If we take the limit,  $N \rightarrow \infty$ , this result is equivalent to the integral

$$W = F_x \int dx + F_y \int dy = \int \mathbf{F} \cdot d\mathbf{s}$$

Thus the work done consists of the sum of the x-component of the force multiplied by the x-component of the displacement and a similar term for the y-components. Fig. 2-2 shows the situation.

As in the one-dimensional case, a sign convention applies in the evaluation of Eq. (1) and similar integrals. Even though the path followed may involve negative components for the displacement, we always write  $d\mathbf{s} = dx \mathbf{i} + dy \mathbf{j}$  in  $\mathbf{F} \cdot d\mathbf{s}$ . By always taking the lower limit to correspond to the initial position and the upper limit to correspond to the final position, the sign associated with the displacement is correctly taken into account. The algebraic sign associated with  $\mathbf{F}$ , which depends on the coordinate frame selection, must, however always be properly included. Observe carefully in the following example how this convention applies.

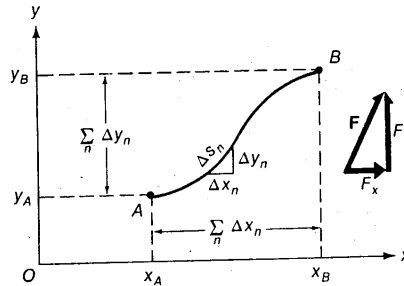


Fig. 2-2: Graphical representation of the work done by a constant force  $F$ .

The three-dimensional generalization of the result in eq. (1) involves the additional term  $F_z (z_B - z_A)$ . The work done by a constant force  $\mathbf{F}$  can then be expressed in vector notation as

$$W = \mathbf{F} \cdot (\mathbf{r}_B - \mathbf{r}_A) = \mathbf{F} \cdot d\mathbf{s}$$

### EXAMPLE

A pendulum bob is displaced from the equilibrium position and is raised to a vertical height  $h$ , as shown in the figure. If the bob is released from this height at rest, what amount of work has been done by the gravitational force when the bob arrives at the bottom of the swing?

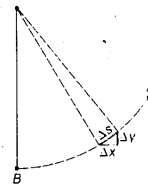
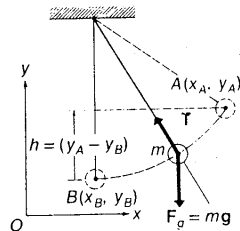


Fig. 2-3

#### Solution :

Because  $\mathbf{F}_g$  is constant, with  $F_{gy} = -mg$  and  $F_{gx} = 0$ , we have from eq. (1), noting that  $y_B - y_A = -h$ .

$$\begin{aligned} W_g &= F_{gx}(x_B - x_A) + F_{gy}(y_B - y_A) \\ &= 0(x_B - x_A) + (-mg)(-h) \\ &= mgh \end{aligned}$$

In this special case, we note that, although  $\mathbf{T}$  is neither constant in direction nor constant in magnitude,  $\mathbf{T}$  does no work on the bob because  $\Delta \mathbf{s}$  and  $\mathbf{T}$  are perpendicular at all times. Therefore, the total work done on the bob is just  $mgh$ .

### 2-6 : Work done in 3 dimensions

For an object moving along a curved path, we can generalize the work kinetic energy theorem by applying it separately to the  $x$ ,  $y$ , and  $z$  components of the motion.

For the motion illustrated in the figure, we will first consider the  $x$ -component of Newton's 2<sup>nd</sup> law,

$$\Sigma F_x = m a_x$$

In a very short time in which the  $x$  coordinate of the object changes by a small distance  $\Delta x$ , the force and acceleration do not change much. In the limit that  $\Delta x \rightarrow 0$ , we may treat the acceleration as if it were constant in this small step, and so eq. (1), applies

$$\Delta(v_x^2) = 2 a_x \Delta x$$

If we replace  $a_x$  by  $\Sigma F_x / m$  and manipulate the result, we have

$$\Sigma F_x \Delta x = \Delta \left( \frac{1}{2} m v_x^2 \right)$$

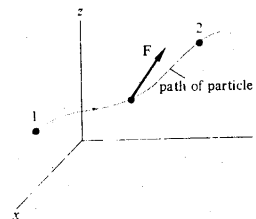


Fig. 2-4: A particle moving along a curved path.

In the same way we obtain

$$\Sigma F_y \Delta y = \Delta (\frac{1}{2} m v_y^2)$$

and

$$\Sigma F_z \Delta z = \Delta (\frac{1}{2} m v_z^2)$$

The sum of these equations can be written as

$$\Sigma F_x \Delta x + \Sigma F_y \Delta y + \Sigma F_z \Delta z = \Delta [\frac{1}{2} m (v_x^2 + v_y^2 + v_z^2)]$$

This equation can be shortened if we use

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

where  $v$  is the instantaneous speed of the particle. The result is

$$\Sigma F_x \Delta x + \Sigma F_y \Delta y + \Sigma F_z \Delta z = \Delta (\frac{1}{2} m v^2) \quad \text{.....(1)}$$

where as before, we identify the quantity  $\frac{1}{2} m v^2$  as the kinetic energy of the particle.

We generalize the definition of the work done by the forces on the particle as it moves the small displacement ( $\Delta x, \Delta y, \Delta z$ ) to be the left hand side of eq. (1).

With this generalization, we conclude that the work done in this small step in the motion is equal to the change in the kinetic energy of the object. For an extended motion, say from point 1 to point 2 on the path of the particle in the figure, the sum of the changes in the kinetic energy in each small step is the net change in the kinetic energy from point 1 to point 2: that is,

$$\Delta K = \Delta (\frac{1}{2} m v^2) = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

By definition, the work done in this extended motion is the sum of the work done in each small step. If we consider each step to be small enough so that we replace  $\Delta x, \Delta y$ , and  $\Delta z$  by the infinitesimal  $dx, dy$ , and  $dz$ , this sum of the work terms is the integral

$$W = \int [\Sigma F_x dx + \Sigma F_y dy + \Sigma F_z dz] \quad \text{.....(2)}$$

With this generalized definition of work, we once again obtain the work-kinetic-energy theorem,

$$W = \Delta (\frac{1}{2} m v^2) = \Delta K \quad \text{.....(3)}$$

## 2-7 : The Work-Energy theorem in one dimension

Consider motion constrained to be along x-axis. And resultant of all forces acting on a particle is a force  $F(x)$ , the Newton's 2<sup>nd</sup> law,

$$F(x) = m d^2 x / dt^2 = m dv / dt \quad \text{.....(1)}$$

Multiplying both sides by  $dx$ ,

$$F(x) dx = dW = m (dv / dt) dx \\ = m (dx / dt) dv = m v dv$$

integrating between the limits,  $x = x_A$  &  $x = x_B$ ,

$$\int_{x_A}^{x_B} dW = \int_{v_A}^{v_B} m v dv \\ \text{or } \int_{x_A}^{x_B} dW = \frac{1}{2} m (v_B^2 - v_A^2) \\ = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2 \\ \text{or } W_{A \rightarrow B} = KE_B - KE_A \quad \text{.....(2)}$$

Eq. (2) shows that the total work done by the sum of all forces acting on the particle during a displacement from A to B is equal to the change in kinetic energy, which is work-energy theorem.

It shows that work is a transitory form of energy; work does not persist. However, the kinetic energy of a particle does persist as long as no forces act on the particle to change its speed. The power of work-energy theorem (eq. 2) lies in the fact that it holds for variable-forces as well as constant forces.



## 2-8 : The Work-Energy Theorem

**Statement :** *The net work done by the forces acting on a particle is equal to the change in the kinetic energy of the particle.*

Consider motion of a particle in several dimensions, and resultant of all forces acting on the particle is a force  $\mathbf{F}(\mathbf{r})$ , then from Newton's 2<sup>nd</sup> law,

$$\mathbf{F} = m \, d\mathbf{v}/dt$$

Integrating the above equation between points a & b,

$$\begin{aligned} \int \mathbf{F} \cdot d\mathbf{r} &= \int m \, d\mathbf{v}/dt \cdot d\mathbf{r} \\ &= \int m \, d\mathbf{r}/dt \, dv \\ &= \int m \, \mathbf{v} \cdot d\mathbf{v} \\ &= m [v^2/2] \\ &= m/2 [v_b^2 - v_a^2] \\ &= \frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2 \\ \text{or } W_{b \rightarrow a} &= KE_b - KE_a \end{aligned} \quad \text{.....(1)}$$

Eq. (1) is the general statement of the work-energy theorem. In this equation, the work done on the particle is by the total force  $\mathbf{F}$ . If  $\mathbf{F}$  is the sum of several forces

$$\mathbf{F} = \Sigma \mathbf{F}_i$$

Then we can write eq. (1) as

$$\Sigma (W_i)_{b \rightarrow a} = \int \mathbf{F}_i \cdot d\mathbf{r} = KE_b - KE_a \quad \text{.....(2)}$$

If instead of a single particle, we consider the center of mass of an extended system moving according to the equation

$$\mathbf{F} = M d\mathbf{V}/dt \quad \text{.....(3)}$$

Integrating eq. (3) w.r.t. position,

$$\int \mathbf{F} \cdot d\mathbf{R} = \frac{1}{2} M V_b^2 - \frac{1}{2} M V_a^2 \quad \text{.....(4)}$$

where  $d\mathbf{R} = \mathbf{V} \, dt$  is the displacement of the center of mass in time  $dt$ .

Eq. (4) is the work-energy theorem for the translational motion of an extended system, i.e. it can include rotational motion.

## 2-9 : Work-Energy Theorem—Proof & Limitations

SEE THE TEXT BOOK

## 2-10 : Power

### Definition :

- i) The rate at which energy is converted from one form of energy to another or the rate at which energy is transported from one place to another.
- ii) The rate at which a force does work on a body is called the power delivered by the force.
- iii) Time rate of doing work.

When we use an automobile engine to move a car up a hill or when we use an electric motor to lift an elevator cage, the important thing is that how much work it can perform in a given amount of time.

If the force does an amount of work  $W$  in an interval of time  $\Delta t$ , then,

$$\text{Average power : } \langle P \rangle = W / \Delta t \quad \text{.....(1)}$$

If we consider small amount of work  $W$  done in the small interval of time  $\Delta t$ , then

$$\text{Instantaneous power : } P_{\text{inst}} = \lim_{\Delta t \rightarrow 0} W / \Delta t \quad \text{.....(2)}$$

In metric system the unit of power is watt, which is the rate of work of one joule per second.

$$1 \text{ watt} = 1 \text{ W} = 1 \text{ J/s} \quad \text{.....(3)}$$

In engineering practice, power is often measured in horse-power ( hp ) units, where  $1 \text{ hp} = 746 \text{ Watt}$



Fig. 2-5: The original definition of the unit horsepower was based on the rate at which horses can raise ~~the~~ water from the depths of coal mines.

Consider a force  $F(x, y, z)$  that is one of the forces acting on a particle. The work done by this force during a general displacement that requires a time  $\Delta t$  is,

$$\Delta W = F_x \Delta x + F_y \Delta y + F_z \Delta z$$

taking the limit  $\Delta t \rightarrow 0$ ,

$$\begin{aligned} dW/dt &= \lim_{\Delta t \rightarrow 0} \Delta W / \Delta t = \lim_{\Delta t \rightarrow 0} (F_x \Delta x / \Delta t + F_y \Delta y / \Delta t + F_z \Delta z / \Delta t) \\ &= F_x dx/dt + F_y dy/dt + F_z dz/dt \\ &= F_x v_x + F_y v_y + F_z v_z \\ &= \mathbf{F} \cdot \mathbf{v} \end{aligned}$$

$$\text{Thus} \quad P = dW/dt = \mathbf{F} \cdot \mathbf{v} \quad \text{.....(4)}$$

Eq. (4) is the definition of power.

## 2-11 : Reference Frames

SEE APPENDIX A &

THE TEXT BOOK

## Chapter 3

### 3-1: Conservative forces

**Conservative force:** A force is conservative if the work done by that force when moving an object from one point to another is independent of the path taken between those two points.

**Conservative forces in one dimension:** In a one-dimensional system a force is conservative if it is a function of position only.

When an object acted on by gravity moves from any position above a zero reference level to any other position, the work of the gravitational force is independent of the path. It is equal to the difference between the final and the initial values of a function, called the gravitational potential energy. If the gravitational force alone acts on the object, the total mechanical energy is conserved, and therefore the gravitational force is called a conservative force.

When an object attached to a spring is moved from one value of the spring extension to any other value, the work of the elastic force is also independent of the path. And it is equal to the difference between the final and initial values of a function called the elastic potential energy. If the elastic force alone acts on the object, the sum of the kinetic and elastic potential energies is conserved, and therefore the elastic force is also a conservative force.

The work of a conservative force has the following properties.

1. It is independent of the path.
2. It is equal to the difference between the final and initial values of a potential energy function.
3. It is completely recoverable.

For a conservative force with frictional force exerted on a moving object by a fixed surface. The work of the frictional force does depend on the path, the longer the path between two given points, the greater the work. There is no function the difference of two values of which equals the work of the frictional force. When we slide an object on a rough fixed surface back to its original position, the frictional force reverses, and instead of recovering the work done in the first displacement, we must again do work on the return trip. So frictional force is not completely recoverable. When the frictional force acts alone, the total mechanical energy is not conserved. The frictional force is therefore called a non-conservative or a dissipative force. So *mechanical energy of a body is conserved only when no dissipative forces act on it.*

### 3-2: Work for a roundtrip

The force of gravity and the force of a spring are conservative forces. If we express the work  $W$  as a difference between two potential energies, then

$$W = U_1 - U_2 \quad \text{..... (1)}$$

For a conservative force, the work done when the particle starts at the point  $x$ , and returns to this same point is necessarily zero, since with  $x_2 = x_1$  eq. (1) gives

$$W = U_1 - U_1 = 0 \quad \text{.....(2)}$$

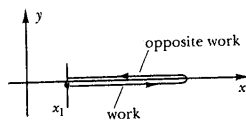


Fig. 3-1: A particle completes roundtrip.

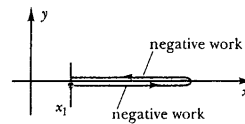


Fig. 3-2: The work is negative for the roundtrip due to friction.

Eq. (2) implies that for a roundtrip that starts and ends at  $x_1$ , the work the force does during the outward portion of the trip is exactly negative of the work the force does during the return portion of the trip, and therefore the net work for the round trip is zero. Thus, the energy supplied by the force is recoverable, as the energy supplied by the force during motion in one direction is restored during the return motion in the opposite direction. For example, when a particle moves downward from some starting point, gravity performs positive work, and when the particle moves upward, returning to its starting point, gravity performs negative work of a magnitude exactly equal to that of the positive work.

The requirement of zero work for a roundtrip can be used to discriminate between conservative and non-conservative forces. Friction is an example of non-conservative force. If we slide a metal block through some distance along a table, and then slide the block back to its starting point, the net work is not zero (fig. 3-2). The work performed by the frictional force during the outward portion of the motion is negative, and the work performed by the frictional force during the return portion of the trip is also negative, as the friction always opposes the motion. And the work done by the frictional force is always negative. Thus, the work done by the frictional force cannot be expressed as a difference between two potential energies, and we cannot formulate a law of conservation of mechanical energy if frictional forces are acting.

If several conservative forces simultaneously act on a particle, then the net potential energy is the sum of all potential energies of all these forces. The total mechanical energy is the sum of the kinetic energy and the net potential energy.

### 3-3: Potential Energy of a spring

The force from a spring acts along a single direction. Let's fix one end of the spring and place an X-axis along the length of the spring so that the spring's free end is at  $x = 0$ . When the free end is displaced by  $x$ , then from Hooke's law, the restoring force is given by

$$\mathbf{F} = -kx \mathbf{i} \quad \dots(1)$$

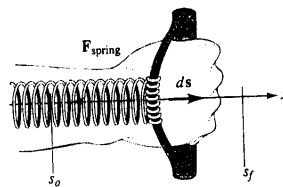


Fig. 3-3: The elastic potential energy of the spring increases with extension

Eq. (1) implies that the force  $\mathbf{F}$  is a function of position only.

Let's calculate the potential energy function for such a spring when we stretch it from  $s_0$  to  $s_f$  as shown in fig. 3-3. The force exerted by the spring is

$$\mathbf{F} = -kx \mathbf{i} \quad \dots (2)$$

If  $x = 0$  at point where the end of the spring rests when unstretched.

The displacement vector is

$$d\mathbf{s} = +ds \mathbf{i} = dx \hat{\mathbf{i}} \quad \dots (3)$$

Calculating the change in potential energy,

$$\Delta U = -W_{\text{spring}}$$

$$= -\int_{s_0}^{s_f} (-kx \mathbf{i}) \cdot (dx \mathbf{i})$$

$$= +\int kx dx$$

$$= +k \int x dx = k \left[ \frac{x^2}{2} \right]_{s_0}^{s_f} = \frac{1}{2} k x_f^2 - \frac{1}{2} k x_0^2$$

$$\text{or } U_f - U_0 = +\frac{1}{2} k s_f^2 - \frac{1}{2} k s_0^2 \quad \dots (4)$$

If we perform the above calculation for a compression of the spring, we obtain the same expression for the potential energy. Thus for a spring, which obeys Hooke's law, the potential energy can be written as

$$U = +\frac{1}{2} k s^2 \quad \dots (5)$$

Where  $s$  is the distance by which the spring is stretched or compressed from its natural length. The potential energy function for such a spring is thus a parabola like that shown in the fig.

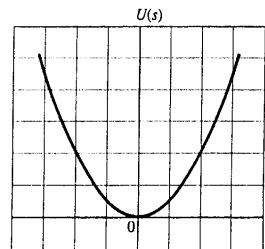


Fig. 3-4: The graph of the potential energy function of a spring

### 3-4: Conservation of mechanical energy

Let's redefine the work performed by a conservative force in terms of the change in potential energy,

$$\Delta U = -W_{\text{cons}} \quad \text{..... (1)}$$

where  $-W_{\text{cons}}$  is the work done by a conservative force.

From work-energy theorem (Section 2-7), we have

$$W_{\text{net}} = \Delta K \quad \text{..... (2)}$$

For n different forces,

$$W_1 + W_2 + W_3 + \dots + W_n = \Delta K$$

If each of these n force is conservative, from eq. (1) we have

$$-\Delta U_1 - \Delta U_2 - \Delta U_3 - \dots - \Delta U_n = \Delta K$$

$$\text{or } \Delta K + \Delta U_1 + \Delta U_2 + \Delta U_3 + \dots + \Delta U_n = 0$$

$$\text{or } \Delta K + \Sigma \Delta U_i = 0 \quad \text{..... (3)}$$

simply putting  $\Sigma \Delta U_i = \Delta U$ , then we have

$$\Delta K + \Delta U = 0 \quad \text{..... (4)}$$

$$\text{or } \Delta(K + U) = 0 \quad \text{..... (5)}$$

where  $(K + U)$  is called the total mechanical energy.

Writing eq. (5) in the form,

$$E = K + U = \text{constant} \quad \text{..... (6)}$$

We can also write eq.(5) in the form, [ as we interpret  $\Delta x = x_f - x_i$  ]

$$(K_f + U_f) - (K_i + U_i) = 0 \quad \text{..... (7)}$$

From eq. (6) we have the statement of Conservation of mechanical energy theorem.

*If only conservative forces perform work and within a system of masses, the total mechanical energy of the system is conserved.*

### 3-5: Derivative of Potential Energy

In section 3-3, we obtain potential energy by integrating the conservative force function. Now we will do the reverse. As we obtained the potential energy from the negative integral of the force function, we obtain the conservative force from the negative derivative of the potential energy function. The potential energy is given for the gravitational force as,

$$U = mgh \quad \text{..... (1)}$$

The component of force in vertical direction is

$$F_y = d/dy (mgh) = -mg \quad \text{..... (2)}$$

Here the negative sign means that the direction of force is downward and potential energy expressed in eq. (1) is upward.

Also for elastic potential energy,

$$U = \frac{1}{2} kx^2 \quad \text{..... (3)}$$

And the force exerted by the spring will be

$$F_y = d/dx (\frac{1}{2} kx^2) = -kx \quad \text{..... (4)}$$

For one-dimensional potential energy function  $U$  of position  $x$  can be written in general case as,

$$F(x) = -dU / dx \quad \text{..... (5)}$$

### 3-6: Simple Harmonic Oscillator

To calculate the kinetic energy and potential energy in case of simple harmonic oscillator, we have velocity  $v$ , from my book "Waves and Oscillations", eq. ( ) of section

$$x = A \cos(\omega t) \quad \dots (1)$$

$$v = -\omega A \sin(\omega t) \quad \dots (2)$$

And kinetic energy is

$$K = \frac{1}{2} m v^2 \quad \dots (3)$$

From eqs. (2) & (3), we have

$$K = \frac{1}{2} m [-\omega A \sin(\omega t)]^2 \quad \dots (4)$$

$$\text{Or } K = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t)$$

$$\text{Or } K = \frac{1}{2} k A^2 \sin^2(\omega t) \quad [m \omega^2 = k] \quad \dots (5)$$

And elastic potential energy is,

$$U = \frac{1}{2} k x^2 \quad \dots (6)$$

From eqs. (1) & (6), we have

$$U = \frac{1}{2} k [A \cos(\omega t)]^2$$

$$\text{Or } U = \frac{1}{2} k A^2 \cos^2(\omega t) \quad \dots (7)$$

From the conservation of energy theorem [eq. (5) of section 3-4]

$$E = K + U \quad \dots (8)$$

From eqs. (5), (7) & (8) we have

$$E = \frac{1}{2} k A^2 \sin^2(\omega t) + \frac{1}{2} k A^2 \cos^2(\omega t)$$

$$\text{Or } E = \frac{1}{2} k A^2 [\sin^2(\omega t) + \cos^2(\omega t)]$$

$$\text{Or } E = \frac{1}{2} k A^2 \quad \dots (9)$$

Eq. (9) shows that the energy of the motion is constant and is proportional to the square of the amplitude of oscillation.

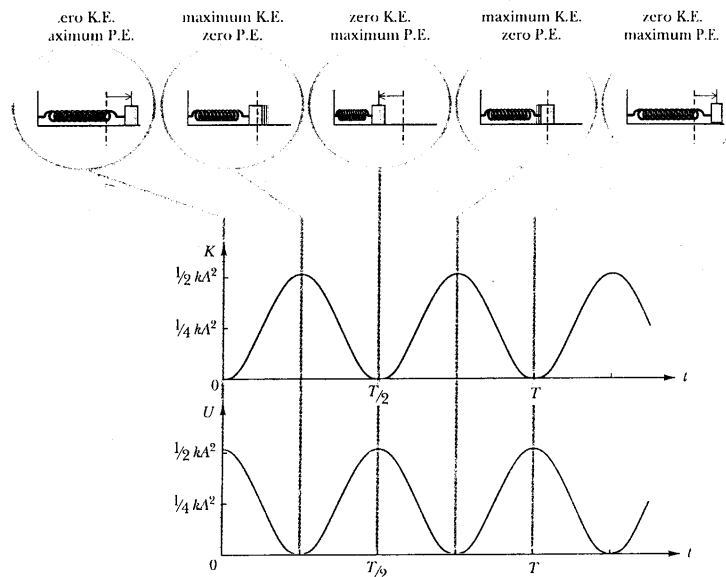


Fig. 3-5: Kinetic energy and potential energy of a simple harmonic oscillator as a function of time.



### 3-7: Conservative Systems—one dimensional solution

SEE THE TEXTBOOK

### 3-8: Potential energy function in 3-dimensional motion

From eq. (5) of section 3-5, we have

$$\mathbf{F} \cdot d\mathbf{x} = -dU \quad \text{..... (1)}$$

Writing eq. (1) in 3-dimensions for the position vector  $\mathbf{r}$ ,

$$\mathbf{F} \cdot d\mathbf{r} = -dU \quad \text{..... (2)}$$

From eq. (4) of section 3-4, we have

$$dK = -dU \quad \text{..... (3)}$$

From eq. (6) of section 3-4, we have

$$\frac{1}{2}mv^2 + U = E \quad \text{..... (4)}$$

For a non-conservative force the work is not an exact differential and is not equal to  $-dU$ . An example of a non-conservative force is force of friction.

When non-conservative forces are present, we can write total force as the sum of  $(\mathbf{F} + \mathbf{F}')$  where  $\mathbf{F}$  is conservative force and  $\mathbf{F}'$  is non-conservative force.

From eqs. (2) & (3), we have

$$dK = \mathbf{F} \cdot d\mathbf{r} + \mathbf{F}' \cdot d\mathbf{r} = -dU + \mathbf{F}' \cdot d\mathbf{r}$$

$$\text{or } dK + dU = \mathbf{F}' \cdot d\mathbf{r}$$

$$\text{or } d(K + U) = \mathbf{F}' \cdot d\mathbf{r} \quad \text{..... (5)}$$

From eq. (5), we see that the quantity  $(K + U)$  is not constant, but increases or decreases as the particle moves on the sign of  $\mathbf{F}' \cdot d\mathbf{r}$ . In case of dissipative forces the direction of  $\mathbf{F}'$  is opposite to that of  $d\mathbf{r}$ , hence  $\mathbf{F}' \cdot d\mathbf{r}$  is negative and the total energy  $(K + U)$  diminishes as the particle moves.

### 3-9: The gradient

From eq. (2) of last section, we have

$$\mathbf{F} \cdot d\mathbf{r} = -dU \quad \dots (1)$$

Applying the rectangular coordinates,

$$F_x dx + F_y dy + F_z dz = -(\partial U / \partial x)dx - (\partial U / \partial y)dy - (\partial U / \partial z)dz$$

It implies that

$$F_x = -(\partial U / \partial x), F_y = -(\partial U / \partial y), F_z = -(\partial U / \partial z) \quad \dots (2)$$

Eq. (2) states that, if the force is conservative, then the components of the force are given by the negative partial derivatives of a potential energy function.

Writing  $\mathbf{F}$  in vector form,

$$\mathbf{F} = -\mathbf{i}(\partial U / \partial x) - \mathbf{j}(\partial U / \partial y) - \mathbf{k}(\partial U / \partial z) \quad \dots (3)$$

Introducing vector differentiation operator

$$\nabla = \mathbf{i}(\partial / \partial x) + \mathbf{j}(\partial / \partial y) + \mathbf{k}(\partial / \partial z) \quad \dots (4)$$

From eqs. (3) & (4), we have

$$\mathbf{F} = -\nabla U = -\text{grad } U \quad \dots (5)$$

Total differential of  $U(x, y, z)$

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\begin{aligned} & \left( \frac{\partial U}{\partial x} \hat{i} + \frac{\partial U}{\partial y} \hat{j} + \frac{\partial U}{\partial z} \hat{k} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \end{aligned}$$

We define

$$\text{grad } U = \frac{\partial U}{\partial x} \hat{i} + \frac{\partial U}{\partial y} \hat{j} + \frac{\partial U}{\partial z} \hat{k}$$

$$\text{So } dU = d\vec{r} \cdot (\text{grad } U)$$

Where  $\nabla$  is called the del operator. The name "gradient" is well chosen for this operator because it does indeed tell us about the "steepness" of the function on which it operates. Mathematically, the gradient of a function is a vector that represents the spatial derivative of the function in direction and magnitude. Physically, the negative gradient of the potential energy function gives the direction and magnitude of the force that acts on a particle located in a field created by other particles. The meaning of the negative sign is that the particle is urged to move in the direction of decreasing potential energy rather than in the opposite direction. An illustration of the gradient is shown in fig. 3-6. Here the potential function is plotted out in the form of contour lines representing the curves of constant potential energy. The force at any point is always normal to the equipotential curve or surface passing through the point in question.

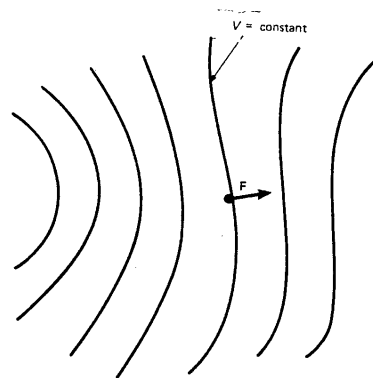


Fig. 3-6: A force field represented by contour lines of potential energy.

### 3-10: Conservative forces in 2-dimensions

Fig. 3-7 (a) is a diagram of the potential energy function that an alpha particle experiences when it is launched at a gold nucleus. It is an ice cube moves if it is launched up the side of this “potential hill”. The actual track of the alpha particle is that observed from directly overhead as in fig. 3-7 (b). Eq. (3) of last section is a detailed mathematical prescription for how to obtain the steepness of a potential hill at any point  $(x, y, z)$ . Its application is illustrated in example 3.1.

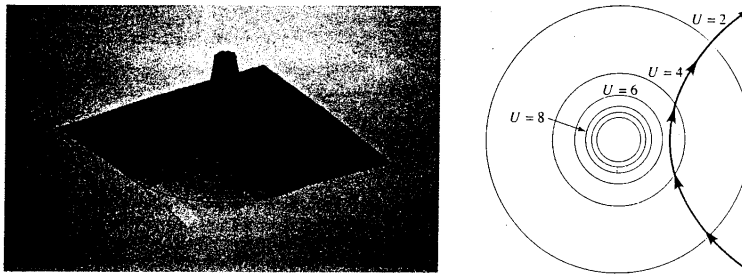


Fig. 3-7 (a) : The potential energy function of a alpha particle from the nucleus of a gold atom. (b) The alpha particle follows hyperbolic path. Circles represent contour lines along which the potential energy is constant.

#### Example 3.1:

For motion restricted to the  $xy$ -plane, the potential energy function ( the side of the potential hill) shown in fig. 3-7 can be described by

$$U(x, y) = A / \sqrt{(x^2 + y^2)}$$

Where  $A$  is a constant. Show that forces calculated from eq. (3) of Section 3-9 agree with your intuition. Specifically, show that the force always points away from the origin and becomes larger as you get closer to the origin.

#### Solution:

We have

$$\begin{aligned} \partial U / \partial x &= \partial / \partial x [A(x^2 + y^2)^{-3/2}] \\ &= -\frac{1}{2} A(x^2 + y^2)^{-3/2} (2x) \\ &= -A x / r^3, \quad [r = \sqrt{(x^2 + y^2)}] \end{aligned}$$

similarly

$$\partial U / \partial y = -A y / r^3$$

and

$$\partial U / \partial z = 0$$

putting these values in the equation,

$$\mathbf{F} = -\mathbf{i} (\partial U / \partial x) - \mathbf{j} (\partial U / \partial y) - \mathbf{k} (\partial U / \partial z)$$

$$\text{Or } \mathbf{F} = +A (\mathbf{x}\mathbf{i} + \mathbf{y}\mathbf{j}) / r^3 = A \mathbf{r} / r^3 \quad \dots (1)$$

The above equation shows that  $\mathbf{F}$  is proportional to  $1/r^2$ ,

as  $r$  becomes small the force increases rapidly,

as the direction of  $\mathbf{F}$  is parallel to the direction of  $\mathbf{r}$ ,

the force points directly “downhill” as we see in fig. 3-7.

### 3-11: Conservative forces in 3-dimensions

If a force is a function of position only, it can be written

$$\mathbf{F} = F_x(x, y, z) \mathbf{i} + F_y(x, y, z) \mathbf{j} + F_z(x, y, z) \mathbf{k} \quad \dots (1)$$

A force with this form is conservative if the following three partial derivative equalities are true

$$\partial F_x / \partial y = \partial F_y / \partial x \quad \dots (2)$$

$$\partial F_y / \partial z = \partial F_z / \partial y \quad \dots (3)$$

$$\partial F_z / \partial x = \partial F_x / \partial z \quad \dots (4)$$

A conservative force is the gradient of a potential function, and should satisfy equations (2), (3) & (4).

From eq. (5) of Section 3-9, we have

$$\mathbf{F} = -\text{grad } U \quad \dots (5)$$

For x-axis, we have

$$F_x = -\partial U / \partial x \quad \dots (6)$$

From eqs. (2) & (6), we have

$$\partial F_x / \partial y = -\partial^2 U / \partial y \partial x \quad \dots (7)$$

$$\text{Also } F_y = -\partial U / \partial y \quad \dots (8)$$

$$\text{And } \partial F_y / \partial x = -\partial^2 U / \partial x \partial y \quad \dots (9)$$

From eqs. (2), (7) & (9) we get

$$\partial^2 U / \partial y \partial x = -\partial^2 U / \partial x \partial y \quad \dots (10)$$

Since order of the differentiation does not matter in eq. (10), so we have proved the equality of eq. (2). In the same manner we can prove the equalities of eqs. (3) & (4).

#### Example 3.2:

Show that the following force function describe a conservative force.

$$\mathbf{F} = xyz^3 \mathbf{i} + \frac{1}{2} x^2 z^3 \mathbf{j} + \frac{3}{2} x^2 y z^2 \mathbf{k} \quad \dots (1)$$

**Solution:**

From first pair of partial derivative equality [eq. (2) of Section 3-11]

$$\partial F_x / \partial y = \partial(xyz^3) / \partial y = xz^3 \quad \dots (2)$$

$$\partial F_y / \partial x = \partial / \partial x (x^2 z^3 / 2) = x z^3 \quad \dots (3)$$

And second pair of partial derivative equality [eq. (3) of Section 3-11]

$$\partial F_y / \partial z = \partial / \partial z (x^2 z^3 / 2) = 3 x^2 z^2 / 2 \quad \dots (4)$$

$$\partial F_z / \partial y = \partial / \partial y (3 x^2 y z^2 / 2) = 3 x^2 z^2 / 2 \quad \dots (5)$$

Finally third pair of partial derivative equality [eq. (4) of Section 3-11]

$$\partial F_x / \partial z = \partial / \partial z (x y z^3) = 3 x y z^2 \quad \dots (6)$$

$$\partial F_z / \partial x = \partial / \partial x (3 x^2 y z^2 / 2) = 3 x y z^2 \quad \dots (7)$$

From eqs. (2) & (3), (4) & (5) and (6) & (7), we see that they are satisfying the three partial derivative equality equations, therefore the force is conservative.

### 3-12: Energy conservation and Isolated Systems

According to the law of conservation of mechanical energy , *if only conservative forces perform work within an isolated system of masses, the total mechanical energy of the system is conserved.*

A system of masses is *isolated only if the masses do not interact with other masses outside the system.* We can sometimes apply the conservation of mechanical energy to non-isolated systems, if

- 1) the interacting masses outside the system are so large that their energies do not change appreciably.
- 2) or the forces connecting the system to the outside masses are so small that no appreciable energy changes are brought about by them.

Consider an example of a ball falling towards the earth. In this case we make

$$mgh = \frac{1}{2} mv^2 \quad \dots (1)$$

In the above equation, we are actually making an approximation. As the ball falls toward the earth, the earth also falls toward the ball. So, strictly speaking, we must include both of these objects in our energy expression,

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} m_e v_e^2 \quad \dots (2)$$

In eq. (2),  $mgh$  is the potential energy of the combined ball-earth system. The mass of the earth is hidden in the factor  $g$ . The kinetic energy gained by the earth is much smaller than the energy gained by the ball. As the kinetic energy gained by the earth is immeasurably small, we simply left it out in this example.

## Chapter 4

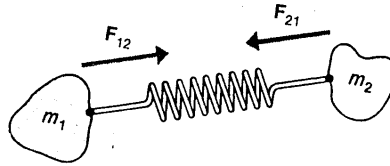
### 4-1: System of Particles

We can consider translational motion of a body that moves as a whole without simultaneous change of its orientation. If a body is rotating, it can be ignored, if we are referring a particular point. If the dimensions of the body is small as compared with the distance we are considering, then such bodies are known as particles. A particle may not be very small as we usually think. For example, in solar system, planets may be treated as particles, as their dimensions are very small as compared with the distances from the sun.

### 4-2: Two Particle Interacting System

Consider two isolated particles coupled together by a rigid spring. Their masses are  $m_1$  and  $m_2$  and they interact with each other. As shown in the figure.

Fig. 4-1:



Considering,  $F_{12}$  is the force exerted on  $m_1$  by  $m_2$ , and  $F_{21}$  is the force exerted on  $m_2$  by  $m_1$ . We have from Newton's 2<sup>nd</sup> Law,

$$F_{12} = m a_1 = m \frac{d}{dt}(v_1) = \frac{d}{dt}(m v_1) = \frac{d}{dt}(p_1) \quad \dots (1)$$

$$\text{and } F_{21} = m a_2 = m \frac{d}{dt}(v_2) = \frac{d}{dt}(m v_2) = \frac{d}{dt}(p_2) \quad \dots (2)$$

Also from Newton's 3<sup>rd</sup> Law;

$$F_{12} = -F_{21} \quad \dots (3)$$

$$\text{Or } F_{12} + F_{21} = 0$$

From above equations we have

$$\frac{d}{dt}(p_1) + \frac{d}{dt}(p_2) = \frac{d}{dt}(p_1 + p_2) = 0$$

integrating the above equation gives,

$$p_1 + p_2 = \text{constant} \quad \dots (4)$$

the above equation is the statement of the conservation of linear momentum. It states that in interacting isolated system of particles has a total system linear momentum that is conserved, i.e., is a constant of the motion, in any inertial frame of reference. The precise value of the constant as measured in two inertial frames depends on the relative velocity of the frames.

### 4-3: Many Particle Systems

It is often useful to identify a certain collection of objects as a system and to refer to all other objects as external things. The choice is arbitrary for taking things in the system or external.

A system has many properties that are analogous to those of a single object. In the figure 4-2, the imagined dashed enclosure about the three particles makes a system.

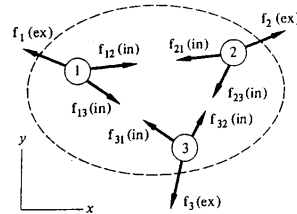


Fig. 4-2:

Applying Newton's 2<sup>nd</sup> law for each particle gives,

$$\Sigma \mathbf{F}_1 = \mathbf{f}_1(\text{ex}) + \mathbf{f}_{12}(\text{in}) + \mathbf{f}_{13}(\text{in}) = m_1 \mathbf{a}_1 \quad \dots (1)$$

$$\Sigma \mathbf{F}_2 = \mathbf{f}_2(\text{ex}) + \mathbf{f}_{21}(\text{in}) + \mathbf{f}_{23}(\text{in}) = m_2 \mathbf{a}_2 \quad \dots (2)$$

$$\& \quad \Sigma \mathbf{F}_3 = \mathbf{f}_3(\text{ex}) + \mathbf{f}_{31}(\text{in}) + \mathbf{f}_{32}(\text{in}) = m_3 \mathbf{a}_3 \quad \dots (3)$$

where  $\mathbf{f}_{ij}(\text{in})$  are internal forces that the particle exert on each other, and  $\mathbf{f}_{ij}(\text{ex})$  are external forces exerted on the particles, which are not part of the system.

Adding equations (1), (2) & (3), we have

$$\mathbf{f}_1(\text{ex}) + \mathbf{f}_{12}(\text{in}) + \mathbf{f}_{13}(\text{in}) + \mathbf{f}_2(\text{ex}) + \mathbf{f}_{21}(\text{in}) + \mathbf{f}_{23}(\text{in}) + \mathbf{f}_3(\text{ex}) + \mathbf{f}_{31}(\text{in}) + \mathbf{f}_{32}(\text{in}) = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 + m_3 \mathbf{a}_3 \quad \dots (4)$$

Since from Newton's 3<sup>rd</sup> law we have

$$\mathbf{f}_{12} + \mathbf{f}_{21} = 0, \quad \mathbf{f}_{13} + \mathbf{f}_{31} = 0 \quad \& \quad \mathbf{f}_{23} + \mathbf{f}_{32} = 0 \quad \dots (5)$$

From eqs. (4) & (5) we get

$$\Sigma \mathbf{F}_{\text{ext}} = \mathbf{f}_1(\text{ex}) + \mathbf{f}_2(\text{ex}) + \mathbf{f}_3(\text{ex}) = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 + m_3 \mathbf{a}_3 \quad \dots (6)$$

Generalizing eq. (4.6) for  $n$  number of particles,

$$\Sigma \mathbf{F}_{\text{ext}} = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 + m_3 \mathbf{a}_3 + \dots + m_n \mathbf{a}_n \quad \dots (7)$$

Considering a special case when all the objects of the system have same acceleration  $\mathbf{a}$ , Then the above equation gives,

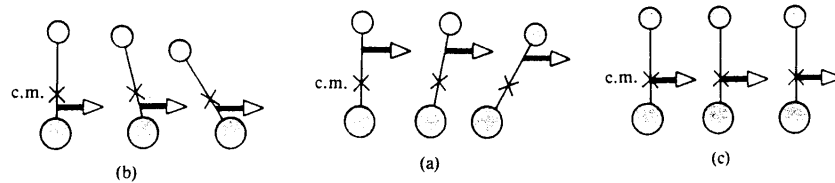
$$\Sigma \mathbf{F}_{\text{ext}} = (m_1 + m_2 + m_3 + \dots + m_n) \mathbf{a} = M \mathbf{a} \quad \dots (8)$$

where  $M$  is the total sum of the masses of all the objects in the system.

#### 4-4: Center of Mass

We can describe the overall motion of a mechanical system in terms of a special point called the center of mass of the system.

Fig. 4-3:



Consider a mechanical system consisting of a pair of particles connected by a light rigid rod. The center of mass (C.M.) is located somewhere on the line joining the particles. If a single force is applied at some point the system will rotate in clockwise or counter-clockwise direction. If force is applied at C.M., the system will not rotate and the rod will move parallel to itself.

The center of mass of the pair of particles described in fig. 4-4 is located on the X-axis and lies somewhere between the particles. The  $x$  coordinate of C.M. is

$$x_c = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \dots (1)$$

For more than two particles,  
we define center of mass vector  $\mathbf{R}$   
of  $n$  particles as,

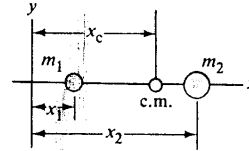


Fig. 4-4:

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3 + \dots + m_n \mathbf{r}_n}{m_1 + m_2 + m_3 + \dots + m_n} \dots (2)$$

or  $\mathbf{R} = 1/M \sum m_i \mathbf{r}_i$  ..... (3)  
where  $M = \sum m_i$

Eq. (3) is a vector equation, representing three separate equations, namely,

$$\begin{aligned} X &= 1/M \sum m_i r_{ix} \\ Y &= 1/M \sum m_i r_{iy} \\ Z &= 1/M \sum m_i r_{iz} \end{aligned} \dots (4)$$

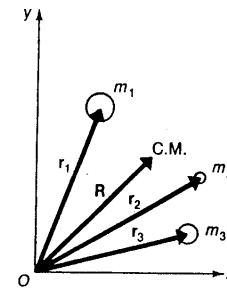


Fig. 4-5:

If the particles are in relative motion, equations (4) will specify center of mass coordinates in a particular instant.

#### 4-5: Uses of Center of Mass

##### a) To determine the motion of a rigid body:

Consider motion of a cricket ball. To calculate the motion of its center of mass, we can use eq. (8) of article 4-3, i.e.,

$$\Sigma \mathbf{F}_{\text{ext}} = M \mathbf{a}$$

although complete motion involves its rotation but its center is fixed.

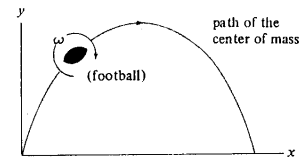


Fig. 4-6: The motion of a cricket ball.

##### b) Motion of two or more particles:

In fig.4-7 it is shown, the motion of two objects rotating in circles about the center of mass of the two-particle system as the center of mass moves with constant velocity to the right.

In fig. 4-7(b) it is shown that how much simpler the motion appears in the frame fixed on the center of mass. We can also apply for the motion of the moon and Earth system.

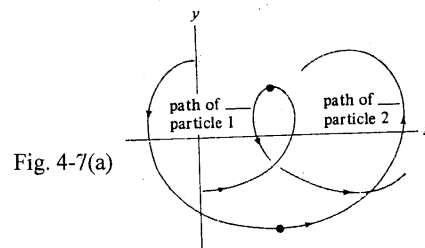
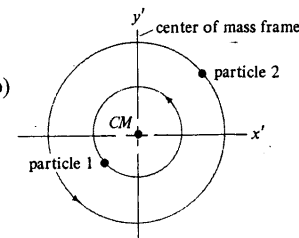


Fig. 4-7(a)

Fig. 4-7(b)



##### c) Motion of two blocks connected by a spring:

Consider two blocks connected by an ideal spring on a frictionless plane. Take block and spring as a system. The external forces, weight of the



system and upward reaction force cancels each other.

So  $\Sigma \mathbf{F}_{\text{ext}} = 0$

$\therefore$  the acceleration  $\mathbf{A}$  of the center of mass is zero.

This means center of mass moves with constant velocity,  $\mathbf{V}_0$ , shown in the fig. 4-8. This value of  $\mathbf{V}_0$  can be determined by how the motion is started.

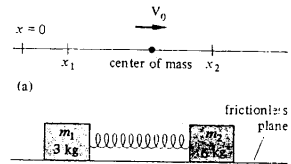


Fig. 4-8:

When a heavy truck collides with a small car, then which vehicle would have more wreckage? How do you decide that the occupants of the car more likely to be injured than those of the truck? It cannot be answered by applying Newton's 2<sup>nd</sup> law, because the force acting between the car and the truck we know very little about. We will find in the preceding discussion that we don't have to know anything about these forces in order to answer the above question.

We will use the concept of momentum and Impulse, and then conservation of momentum.

Consider a particle of constant mass  $m$ . Then from Newton's 2<sup>nd</sup> law

$$\Sigma \mathbf{F} = m \mathbf{a} = m \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m \mathbf{v}) \quad \dots (1)$$

Now we define momentum or linear momentum as;

$$\mathbf{p} = m \mathbf{v} \quad \dots (2)$$

Momentum is a vector quantity, it has a magnitude ( $m \mathbf{v}$ ) and a direction, the same as that of  $\mathbf{v}$ . The heavy truck going 60 km/h has greater magnitude of momentum than a Suzuki car moving with same speed.

From eqs. (1) & (2), we get

$$\Sigma \mathbf{F} = \frac{d}{dt}(\mathbf{p}) \quad \dots (3)$$

Eq. (3) shows that the net force acting on a particle equals the time rate of change of momentum of the particle.

This is not  $\Sigma \mathbf{F} = m \mathbf{a}$ , is the form in which Newton originally stated his 2<sup>nd</sup> law. According to eq. (3), a rapid change in momentum requires a large net force, while a gradual change in momentum requires less net force. This principle is used in the design of automobile safety devices such as air bags. The driver of a fast moving automobile has a large momentum. If the car stops suddenly in a collision, the driver's momentum becomes zero. An air bag causes the driver to lose momentum more gradually than would an abrupt collision with the steering wheel, reducing the force exerted on the driver and the possibility of injury is reduced. The same principle is applied in the cartons of fragile objects for shipping by inserting pads.

#### 4-7: Impulse

We define, the impulse as the integral of force with respect to time,

$$\mathbf{I} = \int \mathbf{F} dt \quad \dots (1)$$

Impulse is a vector quantity. For constant force the above equation reduces to

$$\mathbf{I} = \mathbf{F} \Delta t \quad \dots (2)$$

The above equation shows that for constant force, the impulse is simply product of external force and time interval during which that force acts.

We can calculate impulse from eq. (1), if force is a function of time. If instead, the time dependence of each force component is given in graphical form, then the impulse will be equal to the area under the force curve.

## 4-8: Impulse and Momentum

We have

$$I = \int \Sigma F_{\text{ext}} dt \quad \dots (1)$$

$$\begin{aligned} & \& \Sigma F_{\text{ext}} = m a = m \frac{d}{dt}(v) = \frac{d}{dt}(mv) = \frac{d}{dt}(\mathbf{p}) \\ & \text{or } \Sigma F_{\text{ext}} dt = \frac{d}{dt}(\mathbf{p}) dt \quad \dots (2) \\ & \text{integrating the above equation from } t_1 \text{ to } t_2, \text{ we have} \\ & \int \Sigma F_{\text{ext}} dt = \int \frac{d}{dt}(\mathbf{p}) dt \end{aligned}$$

$$\text{or } \int \Sigma F_{\text{ext}} dt = \int d(\mathbf{p}) = \mathbf{p}_2 - \mathbf{p}_1 = \Delta \mathbf{p} \quad \dots (3)$$

$$\text{or } \int \Sigma F_{\text{ext}} dt = \Delta \mathbf{p} \quad \dots (4)$$

$$\text{or } I = \int \Sigma F_{\text{ext}} dt = \Delta \mathbf{p} \quad \dots (5)$$

$$\text{or } I_{\text{external}} = I_{\text{net}} = \Delta \mathbf{p} \quad \dots (6)$$

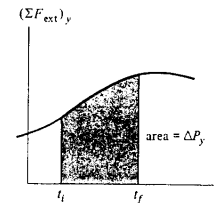


Fig. 4-9:

The Eq. (6) is the **impulse-momentum theorem**, in words, “the net impulse on an object is equal to the object’s change in momentum”.

Taking  $y$  component of eq. (5),

$$I_y = \int \Sigma F_{\text{ext}} dt = \Delta p_y \quad \dots (7)$$

Its graphical representation is given in Fig. 4-9.

The shaded area, which is equal to change in momentum, is the impulse and it is a vector quantity.

## 4-9: Momentum and Kinetic Energy

Let’s see the difference between momentum and kinetic energy. We have the impulse-momentum theorem is:

$$I = \mathbf{p}_2 - \mathbf{p}_1 \quad \dots (1)$$

It says that changes in a body’s momentum are due to impulse, which depends on the time over which the net force acts.

And Work-Energy theorem (section 2-8) is:

$$W_{2 \rightarrow 1} = K_2 - K_1 \quad \dots (2)$$

It says that kinetic energy change when work is done on a body, the total work depends on the distance over which the net force acts.

Consider a body with initial zero velocity, i.e.,  $v_1 = 0$ , then

$$\mathbf{p}_1 = m \mathbf{v}_1 = 0 \quad \& \quad K_1 = \frac{1}{2} m v_1^2 = 0 \quad \dots (3)$$

If a constant net force acts on that body from time  $t_1$  to  $t_2$ . During this interval the body moves a distance  $x$  in the direction of force.

From eqs. (1) & (3), we have

$$\mathbf{p}_2 = \mathbf{p}_1 + I = I \quad \dots (4)$$

So the momentum of the body equals the impulse that accelerated it from rest to its present speed, which depends on the time required for the acceleration.

And kinetic energy of the body at time  $t_2$  is:

$$K_2 = W_{\text{net}} = F s$$

That is, the total work done on the body to accelerate it from rest is the product of the net force and the distance required to accelerate the body.

Example: Select, which ball, you will catch:

First ball:  $m = 1 \text{ kg}$ ,  $v = 5 \text{ m/s}$

Its  $p = m v = 1 \times 5 = 5 \text{ kg.m/s}$  &  $K = \frac{1}{2} m v^2 = \frac{1}{2} \times 1 \times 5^2 = 12.5 \text{ J}$

Second ball:  $m = 0.5 \text{ kg}$ ,  $v = 10 \text{ m/s}$

Its  $p = m v = .5 \times 10 = 5 \text{ kg.m/s}$  &  $K = \frac{1}{2} m v^2 = \frac{1}{2} \times .5 \times 10^2 = 250 \text{ J}$

Both balls have same momentum, but second ball has more kinetic energy. Certainly you will prefer to catch the first ball.

Similarly, for fast moving cricket ball and a bullet, both having same momentum, you will prefer to catch the ball rather than the bullet!

Newton's 2<sup>nd</sup> law is a differential principle; as  $F = m a = d/dt (p)$

But Work-Energy and Impulse-momentum theorems are integral principles;

[See sections 2-8 & 4-8 ].

#### 4-10: Conservation of Momentum

Concept of momentum is important especially in two or more interacting bodies. For any system, the forces that the particles of the system exert on each other are called internal forces. Forces exerted on any part of the system by some object outside it are called external forces.

Consider an isolated system of two astronauts who touch each other as they move in zero gravity in space. Each one will exert a force on the other. The impulses that act on the two bodies will be equal and opposite. Consider these two astronauts as two particles 1 and 2, then from eq. (3) of section 4-6, we have

$$\Sigma F = d/dt (p) \quad \dots (1)$$

$$\text{or } F_{21} = d/dt (p_1) \quad \& \quad F_{12} = d/dt (p_2) \quad \dots (2)$$

The momentum of each particle changes, but these changes are not independent.

From Newton's 3<sup>rd</sup> law,

$$F_{21} = - F_{12} \quad \text{or } F_{21} + F_{12} = 0 \quad \dots (3)$$

from eqs. (2) & (3), we get

$$d/dt (p_1) + d/dt (p_2) = d/dt (p_1 + p_2) = 0 \quad \dots (4)$$

$$\text{taking total momentum} = p = p_1 + p_2 \quad \dots (5)$$

we can generalize the above equation, as

$$p = p_1 + p_2 + p_3 + \dots = m_1 v_1 + m_2 v_2 + m_3 v_3 + \dots \quad \dots (6)$$

$$\text{or total initial momentum} = \text{total final momentum} \quad \dots (7)$$

From eqs. (3) to (6), we get

$$F_{21} + F_{12} = d/dt (p) \quad \dots (8)$$

Eq. (8) states, the time rate of change of total momentum  $p$  is zero. Therefore the total momentum of the system is constant. In case some external forces are present, their vector sum must be zero. So *if the vector sum of the external forces on a system is zero, the total momentum of the system is constant*, which is law of conservation of momentum.

## 4-11: Rocket Propulsion

The driving force for moving carts, boats and automobiles is friction. A car accelerates because the road pushes it and an airplane accelerates because the air pushes it. In space there is no road or air to push against. So the source of propulsion for rocket is different. The rocket motion is based on law of conservation of momentum as applied to a system of particles. Here the system is the rocket plus its ejected fuel.

Momentum considerations are useful for analyzing a system in which the masses of parts and the system changes with time. Here we cannot use Newton's 2<sup>nd</sup> law ( $\mathbf{F} = m \mathbf{a}$ ), because  $m$  is not constant.

A rocket is propelled forward by rearward ejection of burned fuel that initially was in the rocket. The ejected gases acquire some momentum and the rocket receives a compensating momentum in the opposite direction. So the rocket is accelerated from the push of the exhaust gases. The center of the mass of the entire system moves uniformly, independent of the propulsion process.

Consider the rocket motion,

At time  $t$ ,

Mass of the rocket + fuel =  $M + \Delta m$

Velocity =  $v$

After time interval  $\Delta t$ ,

Mass of the rocket + remaining fuel =  $M$

Velocity // // + // // =  $v + \Delta v$

Mass of the exhaust fuel =  $\Delta m$

Velocity of exhaust fuel =  $v - v_{ex}$

From law of conservation of momentum,

Total initial momentum = total final momentum

$(M + \Delta m) v = M(v + \Delta v) + \Delta m(v - v_{ex})$

or  $M \Delta v = v_{ex} \Delta m$  ..... (1)

taking limit,  $\Delta t \rightarrow 0$ ,

then  $\Delta v \rightarrow dv$  &  $\Delta m \rightarrow dm$

And increase in exhaust mass  $dm$ , corresponds an equal decrease in rocket mass, so

$$dm = -dM$$

so eq. (1) gives,

$$M dv = -v_{ex} dM$$

$$\text{or } dv = -v_{ex} dM/M$$

integrating the above equation with limits,

$$\int dv = -v_{ex} \int dM/M$$

where  $M_i$  = initial mass of rocket + fuel

$M_f$  = final mass // // + remaining //

$$\text{or } v_f - v_i = v_{ex} \ln (M_i / M_f)$$

..... (2)

the above equation is the basic expression of rocket propulsion.

Equation (2) shows,

- i) increase in velocity of rocket is proportional to the exhaust velocity,  $v_{ex}$ ,  
i.e., exhaust velocity should be very high.
- ii) increase in velocity is proportional to logarithm of the ratio  $(M_i / M_f)$ ,  
i.e., this ratio should be large or rocket should carry large fuel.

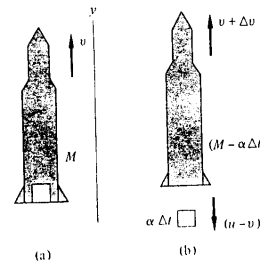


Fig. 4-10:

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## Chapter 5

### 5-1: Collisions

We will use the term collision to represent the event of two particles coming together for a short time, and they produce impulsive forces on each other.

Collision between two bodies, e.g., a car and truck, fast cyclist and a pedestrian, or  $\alpha$ -particle and a heavy nucleus, involves a violent change of motion. This change is brought by strong forces that act suddenly when bodies contact for a short time. The forces that act during a collision usually vary in a rather complicated way.

Consider a collision on an atomic scale. Taking example of the collision of a proton with  $\alpha$ -particle. Since both particles are positively charged, they repel each other due to strong electrostatic forces. It is usually called scattering process.

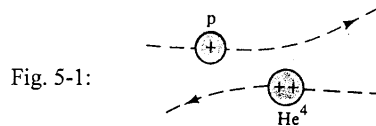


Fig. 5-1:

Consider two particles of mass  $m_1$  and  $m_2$  colliding each other. The impulsive forces may vary in a complicated way as shown in fig. 5-2.

We have from eq. (4) of Section 4-8,

$$\Delta p_1 = \int F_{12} dt \quad \dots (1)$$

$$\text{and} \quad \Delta p_2 = \int F_{21} dt \quad \dots (2)$$

From Newton's 3<sup>rd</sup> law, we have

$$F_{12} = -F_{21} \quad \dots (3)$$

The above relation is shown in Fig. 5-2. Now from eqs. (1), (2) & (3), we get

$$\Delta p_1 = -\Delta p_2$$

$$\text{or} \quad \Delta p_1 + \Delta p_2 = 0$$

The above equation shows that the total change in momentum is zero, so the total momentum of the system is constant.

This is the result if there are no external forces acting on the system. However this result is also valid if we consider the motion just before and after the collision. Since the impulsive forces due to the collision are internal, they do not effect the total momentum of the system. So we conclude for any type of collision, the total momentum of the system just before the collision equals the total momentum of the system just after the collision.

For all collisions, the total momentum is always conserved, but total kinetic energy may or may not be conserved. We can divide collisions into 4 types.

- i) An **elastic collision** is one in which both momentum and kinetic energy are conserved. e.g., billiard ball collisions and collisions of the air molecules with the walls of a container. Elastic collisions occur in atomic and sub-atomic particles, usually they do not occur in macroscopic scale.

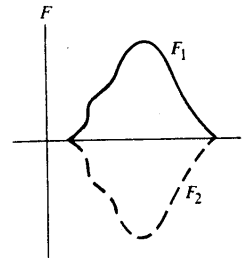


Fig.5-2:

- ii) An **inelastic collision** is one in which only momentum is conserved, kinetic energy is not conserved. e.g., the collision of a rubber ball with a hard surface. Here some of the kinetic energy is lost as the ball is deformed.
- iii) A **perfectly inelastic collision** is an inelastic collision in which the two objects stick together after the collision. e.g., when two wax pieces collide, they stick together and move with some common velocity after the collision.
- iv) A **super-elastic collision** is one in which kinetic energy is larger after the collision than before. Here internal energy of the system is converted to kinetic energy in the collision, for example, from the chemical or nuclear energy released by an explosion occurring at the time of the collision. Such collisions are rarely encountered in everyday world. They are important in atomic and nuclear physics.

## 5-2: Impulse and Collisions

The force that two colliding bodies exert on each other acts only for a short time, giving a brief but strong push, such a force is called an **impulsive force**. During a collision the impulsive force is much stronger than any other forces that may be present. This impulsive force produce a large change in motion while other forces produce very small change. For example, when a car collides with a heavy trailer, the effects produce by gravity and by frictional force of the road during the collision are insignificant.

The Impulse-Momentum relation (section 4-8) has resemblance with Work-Energy relation (section 2-8). Newton's 2<sup>nd</sup> law was used in driving Impulse-Momentum relation. So it applies only for inertial frame of reference. Work-Energy relation involves only the velocity, although Newton's law involves its derivation, the acceleration. Working directly with velocity is easier than finding the position of the body as a function of time, which is acceleration. Another advantage of work-energy relation is that it involves scalar such as  $W$ , while the other involves vector such as  $\mathbf{F}$ , and scalars are easier to handle than vectors. Also the forces in nature usually do not depend upon time, but they depend upon position.

Although impulse has limited applications, still it is useful to think what happens during a collision. Fig. 5-3 illustrates qualitatively the time dependence of the repulsive force acting on two colliding cricket balls.

In fig. 5-4, two table tennis balls have "head-on" collision on an air table.

For simplicity we have taken "head-on" collision so that the force can be represented by a scalar quantity. The direction of the force is defined to be positive. The collision is abrupt because the bodies interact only when they are actually touching. When this happens very strong repulsive contact forces develop for a very short time. If the numerical value of the impulse is same in both cases, that is, if the area under the  $F(t)$  curves in the two figures are equal,

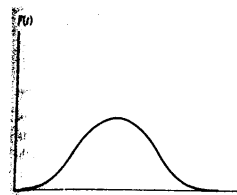


Fig. 5-3:

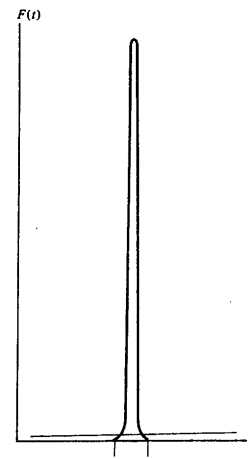
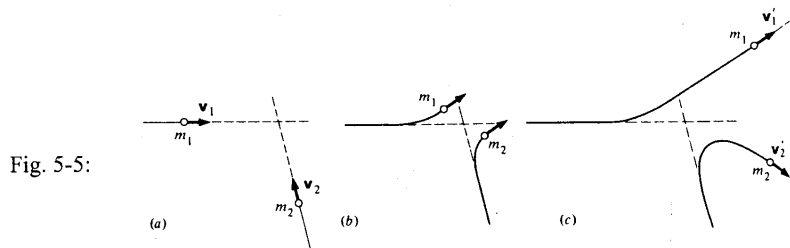


Fig. 5-4

then the momentum transfer will be same in the two collisions. Actually the impulse counts in determining momentum transfer, not the detailed behavior of force  $F(t)$ .

### 5-3: Collisions & Conservative Laws

Here we will see how to apply conservation laws of momentum and energy in scattering experiments. First scattering experiment was performed by Rutherford in 1911, in which  $\alpha$ -particles were scattered by a thin gold foil. A high-energy particle accelerator many miles long have little in common with Rutherford's tabletop experiment. But its purpose is the same, to discover the interaction forces between particles by studying how they scatter.



The above figure shows three stages during the collision of two particles.

- long before collision, each particle is free, as interaction forces are important when particles are close.
- When particles approach, the momentum and energy of each particle change due to interaction of forces.
- After the collision, the particles are again free and move in new directions and velocities.

In this collision we can apply law of conservation of momentum, which is;

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2$$

Although the total energy of the system is always conserved in collisions, part of the kinetic energy may be converted to some other form. So we can write energy equation as;

$$KE_i = KE_f + Q$$

where  $Q$  is amount of kinetic energy converted to another form

$$\text{or } \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2 + Q$$

for most collisions kinetic energy is lost and  $Q$  is positive. In case of super-elastic collisions,  $Q$  is negative, as internal energy of the system is converted into kinetic energy during the collision.



## 5-4: Collisions in one dimension

When two objects collide and the initial and final velocities of both are parallel or anti-parallel the collision is said to be one-dimensional. The collision of two boxcars on a railway track is an example of collisions in one dimension. Generally, the collision of any two bodies that approach head-on and recoil along their original line of motion is one dimensional collision. Although these collisions are exceptional, but they display a simple way of some important features of more complicated collisions.

### i) Perfectly elastic collisions:

In an elastic collision of two particles moving along a straight line, the laws of conservation of momentum and energy completely determine the final velocities in terms of the initial velocities.

If the net external force on the system of masses is zero so that momentum is also conserved, for one-dimensional collision, then from laws of conservation of energy and momentum, we have two equations.

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \quad \dots (1)$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2 \quad \dots (2)$$

from eqs. (1) & (2), we have

$$m_1 (v_1 - v'_1) = m_2 (v'_2 - v_2) \quad \dots (3)$$

$$m_1 (v_1^2 - v'^2_1) = m_2 (v'^2_2 - v_2^2) \quad \dots (4)$$

dividing eq. (4) with eq. (3), we get

$$v_1 + v'_1 = v'_2 + v_2 \quad \dots (5)$$

multiplying eq. (5) with  $m_1$  and then with  $m_2$ , so we get eqs. (6) & (7)

$$m_1(v_1 + v'_1) = m_1(v'_2 + v_2) \quad \dots (6)$$

$$m_2(v_1 + v'_1) = m_2(v'_2 + v_2) \quad \dots (7)$$

Subtracting eq. (3) with eq. (7), we get

$$v'_1 = \frac{(m_1 - m_2)}{m_1 + m_2} v_1 + \frac{2(m_2)}{m_1 + m_2} v_2 \quad \dots (8)$$

Adding eqs. (3) and (6), we get

$$v'_2 = \frac{2(m_1)}{m_1 + m_2} v_1 + \frac{(m_2 - m_1)}{m_1 + m_2} v_2 \quad \dots (9)$$

from eqs. (8) & (9), we can solve for any two unknown quantities.

From eq. (5), we have

$$v_1 - v_2 = v'_2 - v'_1 \quad \dots (10)$$

where  $(v_1 - v_2)$  is the speed of approach of the two objects, and  $(v'_2 - v'_1)$  is the speed of separation. Eq. (10) tells us that for perfectly elastic collisions, the speed of approach is equal to the speed of separation.

### ii) Perfectly Inelastic Collisions:

In inelastic collisions there is some loss of kinetic energy during the collision, so only law of conservation of momentum is applicable. If the collision is totally inelastic, so a maximum amount of kinetic energy is lost, then from the conservation of momentum, we determine the velocities of both the particles after the collision.

In perfectly inelastic collision, the particles do not bounce off each other instead they stick together. For example two boxcars on a railway track that couple together. In

football, when a player tackles a running back, he hopes the collision will be inelastic. Under these conditions, it is clear that the velocity of both particles must coincide with the velocity of the center of mass. The conservation of momentum can be applied to the system composed of  $m_1$  and  $m_2$ .

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v' \quad \dots (11)$$

In the above equation we have five variables. If four of them are known, we can solve for the 5<sup>th</sup> one.

### iii) Partially Elastic Collisions:

Most real collisions are partially elastic collisions. Here the kinetic energy is reduced and the speed of separation is less than the speed of approach. And coefficient of restitution is between zero and one.

We define coefficient of restitution,  $\epsilon$ ,

$$\epsilon = \frac{\text{speed of separation}}{\text{speed of approach}}$$

$$\text{or } \epsilon = \frac{v'_2 - v'_1}{v_1 - v_2}$$

for perfectly elastic collisions;

$$\epsilon = 1$$

// partially elastic //  $0 < \epsilon < 1$

// perfectly inelastic //  $\epsilon = 0$

// super-elastic //  $\epsilon > 1$

## 5-5: Collisions in two dimensions

In two dimensions when we apply law of conservation of momentum, we get two component equations.

Consider a particle of mass  $m_1$  collide with a particle of mass  $m_2$ , and  $m_2$  is initially at rest. In the fig.5-6 the parameter  $b$  is called the impact parameter.

We see if  $b$  is zero, the collision is head-on.

After the collision,  $m_1$  moves at an angle  $\theta$  with respect to the horizontal and  $m_2$  moves at an angle  $\phi$  w.r.t.

horizontal. Applying law of conservation of momentum along horizontal and vertical directions (x- & y-directions), we have

$$m_1 v_{1i} = m_1 v_{1f} \cos \theta + m_2 v_{2f} \cos \phi \quad \dots (1)$$

$$0 = m_1 v_{1f} \sin \theta + m_2 v_{2f} \sin \phi \quad \dots (2)$$

If the collision is elastic, then applying law of conservation of kinetic energy,

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \dots (3)$$

The above three equations can be used for solving some unknown quantity.

### i) Perfectly Inelastic Collisions:

In collisions in two dimensions, we can solve the problems simply by decomposing the problem into two one-dimensional problems and solve each of them. When finished just recombine to find the resultant.

### ii) Perfectly Elastic Collisions:

For perfectly elastic collisions we can apply law of conservation of kinetic energy along with conservation of momentum. We will get three equations to solve three unknown quantities. After the collision we have four unknown quantities; two directions

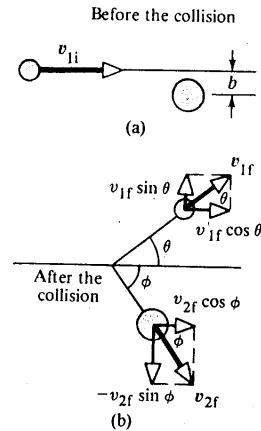


Fig. 5-6:

and two velocities of the particles. To solve this problem we should know at least one direction of final velocity of the particle along with initial velocities and directions.

If you play carom-board or billiard, you must know that the result of two-dimensional collision can range anywhere from head-on to a barely glancing impact. Also the details of how the collision force depends on the separation of the particles during the collision influences the result. In atomic and nuclear physics we see the cases of perfectly elastic collisions.

### iii) Partially Elastic Collisions:

For partially elastic collisions, only momentum is conserved not kinetic energy during the collision. So we can apply only law of conservation of momentum to solve the problem. We will get two equations, and only two unknown quantities can be solved in this case.

## 5-6: Center of Mass

If measurements are made, during a collision experiment, with reference to a frame fixed in the laboratory, such frame is called laboratory frame of reference. If we see the collision experiment from a reference frame attached to the center of mass of the particles then that is called center-of-mass frame of reference. It is usually simpler to treat 2-dimensional or 3-dimensional collision problems in center of mass (CM) frame of reference (or coordinate system) than in laboratory (L) frame of reference.

Consider two particles of masses  $m_1$  and  $m_2$  with velocities  $v_1$  and  $v_2$ . The CM velocity is will be (shown in fig.5-7(a) with  $V$ )

$$V = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

and velocities in CM are;

$$\begin{aligned} v_{1c} &= v_1 - V \\ &= \frac{m_2}{m_1 + m_2} (v_1 - v_2) \end{aligned}$$

and

$$\begin{aligned} v_{2c} &= v_2 - V \\ &= -\frac{m_1}{m_1 + m_2} (v_1 - v_2) \end{aligned}$$

and relative velocity is,

$$v = v_1 - v_2$$

The two momenta in CM are,

$$p_{1c} = m_1 v_{1c} = \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)$$

and

$$p_{2c} = m_2 v_{2c} = -\frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)$$

From the above two relations we see that the total momentum of the CM system is zero. The total momentum of L system is,

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) V$$

since total momentum is conserved in the collision,  $V$  is constant.

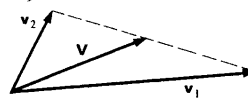
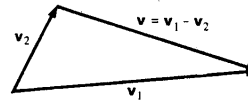
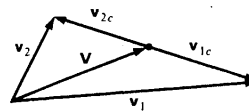


Fig. 5-7:



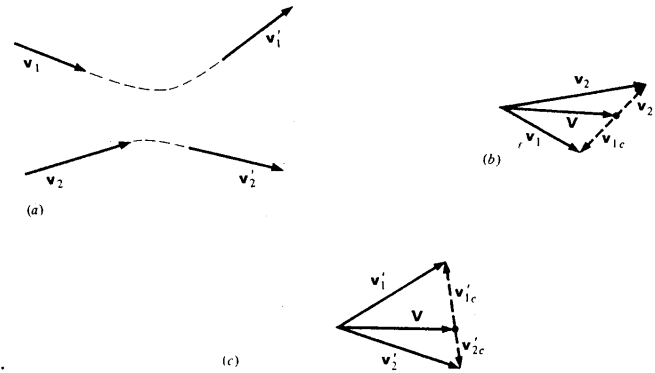


Fig. 5-8:

Fig. 5-8 (a) shows the trajectories and velocities of two colliding particles.

Fig. (b) shows the initial velocities in the L and CM systems. All the vectors lie in the same plane.  $v_{1c}$  and  $v_{2c}$  must be back to back since the total momentum in CM system is zero.

Fig. (c) shows velocities in CM system after the collision. It also shows the final velocities in the L system.

It should be noted that the plane of fig. (a) is not necessarily is that of plane in fig. (c). The geometrical relation between initial and final velocities in the L system is somewhat complicated. In CM system it is simple. A plane known as plane of scattering determines the initial and final velocities in CM system. Each particle is deflected through the same scattering angle  $\Theta$  in this plane. The interaction force must be known in order to calculate  $\Theta$ . Or to calculate interaction force we must know  $\Theta$ .

## Appendix A

### Reference Frames and Relativity

To see things from someone else's point of view is sometimes difficult.

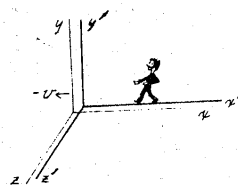
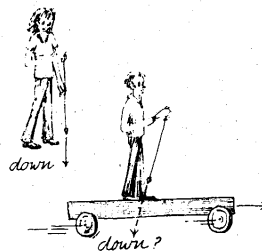
If you pass a slowly moving car, you might notice that the other car is moving backward. The other person in other car might claim that your car is moving forward. Describing events from different reference frames is the subject of **relativity**. That word usually implies Einstein's theories of special and general relativity. The special theory of relativity shows up only when an object is moving nearly the speed of light. The general theory of relativity accounts for what happens in reference frames that are accelerating with respect to our own.

A **Galilean transformation** of coordinates allows us to simplify the description of many events in which objects are moving with respect to us. For instance, we can analyse cases where an object has a constant speed in one direction while undergoing acceleration in a particular direction. For example in describing trajectory of a projectile. Another application is concerning simple harmonic motion. Galilean transformation can simplify the description of colliding objects.

### Galilean Transformation

Two people with measuring instruments, observe the same phenomenon. It might be a ball shooting through the air, or the collision of two objects, or a weight on a spring moving up and down with simple harmonic motion. If the two observers are moving with constant velocity with respect to each other, what differences do they observe in the phenomenon? We will give each observer his own  $x, y, z$  coordinate system. One of them will move in the  $x$ -direction at constant velocity,  $v$ , with respect to the other one.

In the figure,  $S'$  frame is moving to the right with respect to the  $S$  frame. Both are moving with respect to each other. An observer in  $S'$  frame assumes that his frame is stationary while  $S$  frame is moving to the left with a velocity  $-v$ . While the observer in  $S$  frame will look at the matter conversely.



In order to transform a description of motion from one frame to the other, we will write down the Galilean transformation as

$$\begin{aligned} x &= x' + vt' \\ z &= z' \end{aligned} \quad \begin{aligned} y &= y' \\ t &= t' \end{aligned} \quad \dots(1)$$

## Accelerated Reference Frames

If you are sitting in a car and that is racing forward on start off, are you accelerated forward or backward? The observer on the road would say that you are certainly accelerated forward. You yourself know that you are thrown backward—against your seat. If your car is turning around a corner to the left, are you thrown toward the door or in the other direction. An observer watching from a roadway above would conclude that you started to travel along a line tangential to the car's curved path. In both these cases, it appears that the direction of forces and the consequent accelerations depend on the reference frame in which they are described.

Newton's three laws of motion apply only within the reference frame at rest or moving with constant velocity with respect to fixed points. Such a frame is called *inertial reference frame*—a reference frame in which law of inertia hold.

*In accelerated reference frame or non-inertial reference frame is that in which law of inertia does not hold.*

## Special theory of Relativity

### Definitions :

**Special theory of relativity :** The theory applicable to inertial frames of reference.

**General theory of relativity :** The theory which is good for non-inertial frames of reference.

### Postulates of Special theory of relativity :

Special theory of relativity based on Einstein's two postulates:

1. The laws of Physics are the same in all inertial frames of reference.
2. The velocity of light in free space is a constant  $c$  regardless of the state of motion of the source.

### Results of Special theory of relativity

There are four distinct and astonishing results of special theory of relativity.

#### 1. Increase of mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

#### 2. Length contraction

$$l = l_0 \sqrt{1 - v^2/c^2}$$

#### 3. Time dilation

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}}$$

#### 4. Mass-Energy conversion

$$E = mc^2$$

## APPENDIX-B

**Dimension:** A measurement of any sort; especially length, height and width.

**One dimension:** A measurement which needs a single reference point; e.g., a point on a line.

**Two dimensions:** Measurement which needs two references; e.g., a point on a plane (x-y plane).

**Three dimensions:** Measurement which needs three references; e.g., a point in space (x-y-z coordinate system).

**Four dimensions:** Measurement which needs four references; e.g., a point in space + time coordinates (relativistic frame of reference).

**Coordinate:** Suitable sets of numbers with which the positions of points are specified.

**Cartesian coordinate:** Coordinates referred to three mutually perpendicular straight lines.

<u>Dimension</u>	<u>No. of axis</u>	<u>Example</u>	<u>Illustration</u>
One	1 (x)	line	movement of an ant
Two	2(x, y)	plane	movement of a snake
Three	3(x, y, z)	space	movement of a chicken
Four	4(x, y, z, t)	space + time	four-world vector in relativistic mechanics

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## APPENDIX-C

### PARTIAL DIFFERENTIATION :

Definition: If a differentiation of a function of several independent variables be performed with regard to any one of them just as if the others were constants, it is said to be a partial differentiation.

The symbols  $\partial/\partial x$ ,  $\partial/\partial y$ , ..... are used to denote such differentiations, and the expressions  $\partial f/\partial x$ ,  $\partial f/\partial y$ , ..... are called partial differential coefficients with regard to  $x$ ,  $y$ , ..... Respectively.

### Analytical Version :

If we have a function  $f(x, y, z)$ , then

$$\partial f/\partial x = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x, y, z) - f(x, y, z)}{\delta x}$$

### Example :

$$F = x^3 + y^3 - 3axy$$

$$\partial f/\partial x = 3x^2 - 3ay$$

$$\& \quad \partial f/\partial y = 3y^2 - 3ax$$


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